Denotational Design

from meanings to programs

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Abstraction

Abstraction

The purpose of abstraction is not to be vague,

but to create a new semantic level

in which one can be absolutely precise.

- Edsger Dijkstra

Goals

- *Abstractions*: precise, elegant, reusable.
- *Implementations*: correct, efficient, maintainable.
- *Documentation*: clear, simple, accurate.

Not even wrong

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It is not only not right, it is not even wrong.

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What we wish, that we readily believe.

- Demosthenes

Denotative programming

Peter Landin recommended "denotative" to replace ill-defined "functional" and "declarative".

Properties:

- Nested expression structure.
- Each expression *denotes* something,
- depending only on denotations of subexpressions.

"...gives us a test for whether the notation is genuinely functional or merely masquerading." (*The Next 700 Programming Languages*, 1966)

Denotational design

Design methodology for "genuinely functional" programming:

- Precise, simple, and compelling specification.
- Informs *use* and *implementation* without entangling them.
- Standard algebraic abstractions.
- Free of abstraction leaks.
- Laws for free.
- Principled construction of correct implementation.

Overview

- Broad outline:
  - Example, informally
  - *Pretty pictures*
  - Principles
  - More examples
  - Reflection

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- Broad outline:
  - Example, informally
  - *Pretty pictures*
  - Principles
  - More examples
  - Reflection
- Discussion throughout
- Try it on.

Example: image synthesis/manipulation

- How to start?
- What is success?

Functionality

Functionality

- Import & export
- Spatial transformation:
  - Affine: translate, scale, rotate
  - Non-affine: swirls, lenses, inversions, ...
- Cropping
- Monochrome
- Overlay
- Blend
- Blur & sharpen
- Geometry, gradients, ...

API first pass

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type *Image*

$$
\begin{aligned}
&\mathit{over} && :: \mathit{Image} \to \mathit{Image} \to \mathit{Image} \\
&\mathit{transform} && :: \mathit{Transform} \to \mathit{Image} \to \mathit{Image} \\
&\mathit{crop} && :: \mathit{Region} \to \mathit{Image} \to \mathit{Image} \\
&\mathit{monochrome} && :: \mathit{Color} \to \mathit{Image}
\end{aligned}
$$

-- shapes, gradients, etc.

$$
\begin{aligned}
&\mathit{fromBitmap} && :: \mathit{Bitmap} \to \mathit{Image} \\
&\mathit{toBitmap} && :: \mathit{Image} \to \mathit{Bitmap}
\end{aligned}
$$

How to implement?

How to implement?

wrong first question

What to implement?

What to implement?

- What do these operations mean?

What to implement?

- What do these operations mean?
- More centrally: What do the *types* mean?

What is an image?

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Specification goals:

What is an image?

Specification goals:

- Adequate
- Simple
- Precise

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Why these properties?

What is an image?

What is an image?

My answer: assignment of colors to 2D locations.

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How to make precise?

type *Image*

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Model:

$$\mu :: Image \to (Loc \to Color)$$

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What about regions?

What is an image?

My answer: assignment of colors to 2D locations.

How to make precise?

$$\textbf{type } \mathit{Image}$$

Model:

$$\mu :: \mathit{Image} \to (\mathit{Loc} \to \mathit{Color})$$

What about regions?

$$\mu :: \mathit{Region} \to (\mathit{Loc} \to \mathit{Bool})$$

Specifying *Image* operations

$$\begin{aligned}
\mu\ (over\ top\ bot) &\equiv ... \\
\mu\ (crop\ reg\ im) &\equiv ... \\
\mu\ (monochrome\ c) &\equiv ... \\
\mu\ (transform\ tr\ im) &\equiv ...
\end{aligned}$$

Specifying *Image* operations

$$
\begin{aligned}
\mu\ (over\ top\ bot) &\equiv \lambda p \rightarrow overC\ (\mu\ top\ p)\ (\mu\ bot\ p) \\
\mu\ (crop\ reg\ im) &\equiv \lambda p \rightarrow \textbf{if}\ \mu\ reg\ p\ \textbf{then}\ \mu\ im\ p\ \textbf{else}\ clear \\
\mu\ (monochrome\ c) &\equiv \lambda p \rightarrow c \\
\mu\ (transform\ tr\ im) &\equiv \quad \text{-- coming up}
\end{aligned}
$$

$$overC :: Color \rightarrow Color \rightarrow Color$$

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$$

$$overC :: Color \rightarrow Color \rightarrow Color$$

Note compositionality of μ.

Compositional semantics

Make more explicit:

$$\mu\ (over\ top\ bot) \equiv overS\ (\mu\ top)\ (\mu\ bot)$$
$$\mu\ (crop\ reg\ im)\ \equiv cropS\ (\mu\ reg)\ (\mu\ im)$$

$$overS :: (Loc \to Color) \to (Loc \to Color) \to (Loc \to Color)$$
$$overS\ f\ g = \lambda p \to overC\ (f\ p)\ (g\ p)$$

$$cropS :: (Loc \to Bool) \to (Loc \to Color) \to (Loc \to Color)$$
$$cropS\ f\ g = \lambda p \to \textbf{if}\ f\ p\ \textbf{then}\ g\ p\ \textbf{else}\ clear$$

Generalize and simplify

- What about transforming *regions*?
- Other pointwise combinations (lerp, threshold)?

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Generalize:

$$\begin{aligned}
&\textbf{type } \mathit{Image}\ a \\
&\textbf{type } \mathit{ImageC} = \mathit{Image}\ \mathit{Color} \\
&\textbf{type } \mathit{Region}\ \ = \mathit{Image}\ \mathit{Bool}
\end{aligned}$$

Now some operations become more general.

Generalize and simplify

$$transform :: Transform \to Image\ a \to Image\ a$$
$$cond \quad :: Image\ Bool \to Image\ a \to Image\ a \to Image\ a$$

Generalize and simplify

$$transform :: Transform \to Image\ a \to Image\ a$$
$$cond \quad :: Image\ Bool \to Image\ a \to Image\ a \to Image\ a$$

$$lift_0 :: a \to Image\ a$$
$$lift_1 :: (a \to b) \to (Image\ a \to Image\ b)$$
$$lift_2 :: (a \to b \to c) \to (Image\ a \to Image\ b \to Image\ c)$$
$$\ldots$$

Generalize and simplify

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&lift_2 :: (a \to b \to c) \to (Image\ a \to Image\ b \to Image\ c) \\
&\ldots
\end{aligned}
$$

Specializing,

$$
\begin{aligned}
monochrome &= lift_0 \\
over &= lift_2\ overC \\
crop\ r\ im &= cond\ r\ im\ emptyIm \\
cond &= lift_3\ ifThenElse
\end{aligned}
$$

Spatial transformation

$$\mu :: \mathit{Transform} \to ??$$

$$\mu\ (\mathit{transform}\ \mathit{tr}\ \mathit{im}) \equiv ??$$

Spatial transformation

$$\mu :: Transform \to ??$$

$$\mu\ (transform\ tr\ im) \equiv transformS\ (\mu\ tr)\ (\mu\ im)$$

where

$$transformS :: ?? \to (Loc \to Color) \to (Loc \to Color)$$

Spatial transformation

$$\mu :: Transform \to (Loc \to Loc)$$

$$\mu\ (transform\ tr\ im) \equiv transformS\ (\mu\ tr)\ (\mu\ im)$$

where

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Spatial transformation

$$\mu :: Transform \to (Loc \to Loc)$$

$$\mu\ (transform\ tr\ im) \equiv transformS\ (\mu\ tr)\ (\mu\ im)$$

where

$$transformS :: (Loc \to Loc) \to (Loc \to Color) \to (Loc \to Color)$$
$$transformS\ h\ f = \lambda p \to f\ (h\ p)$$

Subtle implications.

Spatial transformation

$$\mu :: Transform \to (Loc \to Loc)$$

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Subtle implications.

What is *Loc*?

Spatial transformation

$$\mu :: Transform \to (Loc \to Loc)$$

$$\mu\ (transform\ tr\ im) \equiv transformS\ (\mu\ tr)\ (\mu\ im)$$

where

$$transformS :: (Loc \to Loc) \to (Loc \to Color) \to (Loc \to Color)$$
$$transformS\ h\ f = \lambda p \to f\ (h\ p)$$

Subtle implications.

What is *Loc*? My answer: continuous, infinite 2D space.

$$\textbf{type}\ Loc = \mathbb{R}^2$$

Why continuous & infinite (vs discrete/finite) space?

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- Transformation flexibility with simple & precise semantics.
- Modularity/reusability/composability:
  - Fewer assumptions, more uses (resolution-independence).
  - More info available for extraction.
- Integration and differentiation: natural, accurate, efficient.

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- Quality/accuracy.
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- Reconcile differing input sampling rates.

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Principle: Approximations/prunings compose badly, so postpone.

See *Why Functional Programming Matters*.

Examples

Pan gallery

Using standard vocabulary

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- We've created a domain-specific vocabulary.
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- Why would we want to?

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  - Tao check.
  - Specification and laws for free, as we'll see.

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- In Haskell, standard type classes.

Monoid

Interface:

```
class Monoid m where
  ε   :: m                  -- "mempty"
  (⊕) :: m → m → m          -- "mappend"
```

Monoid

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$$
\begin{array}{l}
\textbf{class}\ \mathit{Monoid}\ m\ \textbf{where} \\
\quad \varepsilon \quad :: m \qquad\qquad\qquad\quad \text{-- “mempty”} \\
\quad (\oplus) :: m \to m \to m \quad \text{-- “mappend”}
\end{array}
$$

Laws:

$$
\begin{aligned}
a \oplus \varepsilon &\equiv a \\
\varepsilon \oplus b &\equiv b \\
a \oplus (b \oplus c) &\equiv (a \oplus b) \oplus c
\end{aligned}
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Why do laws *matter*?

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Why do laws *matter*? Compositional (modular) reasoning.

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What monoids have we seen today?

Image monoid

Image monoid

instance *Monoid ImageC* **where**

$$\varepsilon = lift_0\ clear$$

$$(\oplus) = over$$

Image monoid

instance *Monoid ImageC* **where**

$$\begin{aligned} \varepsilon &= lift_0\ clear \\ (\oplus) &= over \end{aligned}$$

Is there a more general form on *Image a*?

Image monoid

```
instance Monoid ImageC where
  ε     = lift0 clear
  (⊕) = over
```

Is there a more general form on *Image a*?

```
instance Monoid a ⇒ Monoid (Image a) where
  ε     = lift0 ε
  (⊕) = lift2 (⊕)
```

Image monoid

instance *Monoid ImageC* **where**

$\varepsilon \quad = lift_0\ clear$

$(\oplus) = over$

Is there a more general form on *Image a*?

instance *Monoid a* $\Rightarrow$ *Monoid* (*Image a*) **where**

$\varepsilon \quad = lift_0\ \varepsilon$

$(\oplus) = lift_2\ (\oplus)$

Do these instances satisfy the *Monoid* laws?

Functor

class *Functor* f **where**

$$fmap :: (a \to b) \to (f\ a \to f\ b)$$

Functor

$$\textbf{class } \mathit{Functor}\ f \textbf{ where}$$
$$\mathit{fmap} :: (a \to b) \to (f\ a \to f\ b)$$

For images?

Functor

class *Functor* f **where**

$$fmap :: (a \to b) \to (f\ a \to f\ b)$$

For images?

instance *Functor Image* **where**

$$fmap = lift_1$$

Functor

$$\textbf{class } \mathit{Functor}\ f \textbf{ where}$$
$$\quad \mathit{fmap} :: (a \to b) \to (f\ a \to f\ b)$$

For images?

$$\textbf{instance } \mathit{Functor}\ \mathit{Image} \textbf{ where}$$
$$\quad \mathit{fmap} = \mathit{lift}_1$$

Laws?

Applicative

$$\textbf{class}\ \mathit{Functor}\ f \Rightarrow \mathit{Applicative}\ f\ \textbf{where}$$

$$\mathit{pure}\ \ :: a \to f\ a$$

$$(\mathbin{<\!\!*\!\!>}) :: f\ (a \to b) \to f\ a \to f\ b$$

Applicative

$$\textbf{class}\ \mathit{Functor}\ f \Rightarrow \mathit{Applicative}\ f\ \textbf{where}$$
$$\quad \mathit{pure}\ \ :: a \to f\ a$$
$$\quad (\circledast) :: f\ (a \to b) \to f\ a \to f\ b$$

For images?

Applicative

$$\textbf{class } \mathit{Functor}\ f \Rightarrow \mathit{Applicative}\ f \textbf{ where}$$
$$\quad \mathit{pure} :: a \to f\ a$$
$$\quad (\langle * \rangle) :: f\ (a \to b) \to f\ a \to f\ b$$

For images?

$$\textbf{instance } \mathit{Applicative}\ \mathit{Image} \textbf{ where}$$
$$\quad \mathit{pure} = \mathit{lift}_0$$
$$\quad (\langle * \rangle) = \mathit{lift}_2\ (\$)$$

From *Applicative*, where $(\langle \$ \rangle) = \mathit{fmap}$:

$$\mathit{liftA}_2\ f\ p\ q = f \langle \$ \rangle p \langle * \rangle q$$
$$\mathit{liftA}_3\ f\ p\ q\ r = f \langle \$ \rangle p \langle * \rangle q \langle * \rangle r$$
$$\quad \text{-- etc}$$

Applicative

$$\textbf{class } \mathit{Functor}\ f \Rightarrow \mathit{Applicative}\ f \textbf{ where}$$
$$\quad \mathit{pure} :: a \to f\ a$$
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$$\textbf{instance } \mathit{Applicative}\ \mathit{Image} \textbf{ where}$$
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From *Applicative*, where $(\langle \$ \rangle) = \mathit{fmap}$:

$$\mathit{liftA}_2\ f\ p\ q = f \langle \$ \rangle p \langle * \rangle q$$
$$\mathit{liftA}_3\ f\ p\ q\ r = f \langle \$ \rangle p \langle * \rangle q \langle * \rangle r$$
$$\quad \text{-- etc}$$

Laws?

Instance semantics

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Monoid:

$$\begin{aligned}\mu\ \varepsilon &\equiv \lambda p \to \varepsilon \\ \mu\ (top \oplus bot) &\equiv \lambda p \to \mu\ top\ p \oplus \mu\ bot\ p\end{aligned}$$

Instance semantics

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Functor:

$$\begin{aligned}\mu\ (fmap\ f\ \ im) &\equiv \lambda p \to f\ (im\ p) \\ &\equiv f \circ im\end{aligned}$$

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Applicative:

$$\begin{aligned}\mu\ (pure\ a) &\equiv \lambda p \to a \\ \mu\ (imf <\!\!*\!\!> imx) &\equiv \lambda p \to (imf\ p)\ (imx\ p)\end{aligned}$$

Monad and *Comonad*

```
class Monad f where
  return :: a → f a
  join   :: f (f a) → f a
```

```
class Functor f ⇒ Comonad f where
  coreturn :: f a → a
  cojoin   :: f a → f (f a)
```

Comonad gives us neighborhood operations.

Monoid specification, revisited

Image monoid specification:

$$\begin{aligned}\mu\ \varepsilon &\equiv \lambda p \to \varepsilon \\ \mu\ (top \oplus bot) &\equiv \lambda p \to \mu\ top\ p \oplus \mu\ bot\ p\end{aligned}$$

Monoid specification, revisited

Image monoid specification:

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Instance for the semantic model:

$$\begin{aligned}&\textbf{instance}\ Monoid\ m \Rightarrow Monoid\ (z \to m)\ \textbf{where} \\ &\quad \varepsilon = \lambda z \to \varepsilon \\ &\quad f \oplus g = \lambda z \to f\ z \oplus g\ z\end{aligned}$$

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Refactoring,

$$\begin{aligned}\mu\ \varepsilon &\equiv \varepsilon \\ \mu\ (top \oplus bot) &\equiv \mu\ top \oplus \mu\ bot\end{aligned}$$

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So μ *distributes* over monoid operations

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Refactoring,

$$\begin{aligned}\mu\ \varepsilon &\equiv \varepsilon \\ \mu\ (top \oplus bot) &\equiv \mu\ top \oplus \mu\ bot\end{aligned}$$

So μ *distributes* over monoid operations, i.e., a monoid homomorphism.

Functor specification, revisited

Functor specification:

$$\mu\ (fmap\ f\ \ im) \equiv f \circ \mu\ im$$

Functor specification, revisited

Functor specification:

$$\mu\ (fmap\ f\ \ im) \equiv f \circ \mu\ im$$

Instance for the semantic model:

instance *Functor* $((\rightarrow)\ u)$ **where**

$$fmap\ f\ \ h = f \circ h$$

Refactoring,

$$\mu\ (fmap\ f\ \ im) \equiv fmap\ f\ \ (\mu\ im)$$

So μ is a *functor* homomorphism.

Applicative specification, revisited

Applicative specification:

$$\begin{aligned}\mu\ (pure\ a) &\equiv \lambda p \to a \\ \mu\ (imf <\!\!*\!\!> imx) &\equiv \lambda p \to (\mu\ imf\ p)\ (\mu\ imx\ p)\end{aligned}$$

Applicative specification, revisited

Applicative specification:

$$\begin{aligned}\mu\ (pure\ a) &\equiv \lambda p \to a\\ \mu\ (imf <\!\!*\!\!> imx) &\equiv \lambda p \to (\mu\ imf\ p)\ (\mu\ imx\ p)\end{aligned}$$

Instance for the semantic model:

$$\begin{aligned}&\textbf{instance}\ Applicative\ ((\to)\ u)\ \textbf{where}\\ &\quad pure\ a = \lambda u \to a\\ &\quad fs <\!\!*\!\!> xs = \lambda u \to (fs\ u)\ (xs\ u)\end{aligned}$$

Refactoring,

$$\begin{aligned}\mu\ (pure\ a) &\equiv pure\ a\\ \mu\ (imf <\!\!*\!\!> imx) &\equiv \mu\ imf <\!\!*\!\!> \mu\ imx\end{aligned}$$

So μ is an *applicative* homomorphism.

Specifications for free

Semantic type class morphism (TCM) principle:

> *The instance's meaning follows the meaning's instance.*

That is, the type acts like its meaning.

Every TCM failure is an abstraction leak.

Strong design principle.

Class laws necessarily hold, as we'll see.

Laws for free

$$\boxed{\begin{aligned} \mu\ \varepsilon &\equiv \varepsilon \\ \mu\ (a \oplus b) &\equiv \mu\ a \oplus \mu\ b \end{aligned}} \Rightarrow \boxed{\begin{aligned} a \oplus \varepsilon &\equiv a \\ \varepsilon \oplus b &\equiv b \\ a \oplus (b \oplus c) &\equiv (a \oplus b) \oplus c \end{aligned}}$$

where equality is *semantic*.

Laws for free

$$\boxed{\begin{aligned} \mu\ \varepsilon &\equiv \varepsilon \\ \mu\ (a \oplus b) &\equiv \mu\ a \oplus \mu\ b \end{aligned}} \Rightarrow \boxed{\begin{aligned} a \oplus \varepsilon &\equiv a \\ \varepsilon \oplus b &\equiv b \\ a \oplus (b \oplus c) &\equiv (a \oplus b) \oplus c \end{aligned}}$$

where equality is *semantic*. Proofs:

$$\boxed{\begin{aligned} & \mu\ (a \oplus \varepsilon) \\ \equiv\ & \mu\ a \oplus \mu\ \varepsilon \\ \equiv\ & \mu\ a \oplus \varepsilon \\ \equiv\ & \mu\ a \end{aligned}} \quad \boxed{\begin{aligned} & \mu\ (\varepsilon \oplus b) \\ \equiv\ & \mu\ \varepsilon \oplus \mu\ b \\ \equiv\ & \varepsilon \oplus \mu\ b \\ \equiv\ & \mu\ b \end{aligned}} \quad \boxed{\begin{aligned} & \mu\ (a \oplus (b \oplus c)) \\ \equiv\ & \mu\ a \oplus (\mu\ b \oplus \mu\ c) \\ \equiv\ & (\mu\ a \oplus \mu\ b) \oplus \mu\ c \\ \equiv\ & \mu\ ((a \oplus b) \oplus c) \end{aligned}}$$

Works for other classes as well.

Example: functional reactive programming

See previous talks:

- *The essence and origins of FRP*
- *A more elegant specification for FRP*

Example: uniform pairs

Type:

$$\textbf{data}\ \mathit{Pair}\ a = a :\# a$$

API: *Monoid*, *Functor*, *Applicative*, *Monad*, *Foldable*, *Traversable*.

Example: uniform pairs

Type:

$$\textbf{data}\ \mathit{Pair}\ a = a :\# a$$

API: *Monoid*, *Functor*, *Applicative*, *Monad*, *Foldable*, *Traversable*.

Specification follows from simple & precise denotation.

Uniform pairs — denotation

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Pair is an *indexable* container. What's the index type?

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$$\begin{aligned}
&\mu :: Pair\ a \to P\ a \\
&\mu\ (u :\#\ v)\ False = u \\
&\mu\ (u :\#\ v)\ True = v
\end{aligned}$$

API specification?

Uniform pairs — denotation

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\end{aligned}$$

API specification? Homomorphisms, as usual!

Uniform pairs — monoid

Monoid homomorphism:

$$\begin{aligned} \mu\ \varepsilon &\equiv \varepsilon \\ \mu\ (u \oplus v) &\equiv \mu\ u \oplus \mu\ v \end{aligned}$$

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In this case,

$$\begin{aligned} &\textbf{instance}\ \mathit{Monoid}\ m \Rightarrow \mathit{Monoid}\ (z \to m)\ \textbf{where} \\ &\quad \varepsilon = \lambda z \to \varepsilon \\ &\quad f \oplus g = \lambda z \to f\ z \oplus g\ z \end{aligned}$$

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so

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Implementation: solve for ε and $(\oplus)$ on the left.

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Implementation: solve for ε and $(\oplus)$ on the left. Hint: find μ^{-1}.

Uniform pairs — other classes

Exercise: apply the same principle for

- *Functor*
- *Applicative*
- *Monad*
- *Foldable*
- *Traversable*

Example: streams

$$\textbf{data}\ \mathit{Stream}\ a = \mathit{Cons}\ a\ (\mathit{Stream}\ a)$$

API: same classes as with Pair.

Denotation?

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Denotation? Hint: *Stream* is also an indexable type.

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Denotation? Hint: *Stream* is also an indexable type.

$$\textbf{data}\ S\ a = \mathit{Nat} \to a$$

$$\textbf{data}\ \mathit{Nat} = \mathit{Zero} \mid \mathit{Succ}\ \mathit{Nat}$$

Interpret *Stream* as S:

$$\begin{aligned}
&\mu :: \mathit{Stream}\ a \to S\ a \\
&\mu\ (\mathit{Cons}\ a\ _)\ \ \mathit{Zero} &&= a \\
&\mu\ (\mathit{Cons}\ _\ as)\ (\mathit{Succ}\ n) &&= \mu\ as\ n
\end{aligned}$$

Memo tries

Generalizes *Pair* and *Stream*:

$$\textbf{type } a \twoheadrightarrow b$$

$$\mu :: (a \twoheadrightarrow b) \to (a \to b)$$

API: classes as above, plus *Category*.

Exploit inverses to calculate instances, e.g.,

$$\mu\ id \equiv id$$
$$\Leftarrow id \equiv \mu^{-1}\ id$$

$$\mu\ (g \circ f) \equiv \mu\ g \circ \mu\ f$$
$$\Leftarrow g \circ f \equiv \mu^{-1}\ (\mu\ g \circ \mu\ f)$$

Then simplify/optimize.

Example: lists with a bonus

$$\textbf{data}\ \mathit{ListX}\ a\ b = \mathit{Done}\ b \mid \mathit{Cons}\ a\ (\mathit{ListX}\ a\ b)$$

Denotation:

Example: lists with a bonus

$$\textbf{data}\ \mathit{ListX}\ a\ b = \mathit{Done}\ b \mid \mathit{Cons}\ a\ (\mathit{ListX}\ a\ b)$$

Denotation:

$$\begin{aligned}
&\mu :: \mathit{ListX}\ a\ b \to ([a], b)\\
&\mu\ (\mathit{Done}\ b) &&= ([\,], b)\\
&\mu\ (\mathit{Cons}\ a\ asb) &&= (a : as, b)\ \textbf{where}\ (as, b) = \mu\ asb
\end{aligned}$$

Exercise: instances, including

$$\textbf{instance}\ \mathit{Monad}\ (\mathit{ListX}\ a)\ \textbf{where}\ \ldots$$

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\end{aligned}$$

Exercise: instances, including

$$\textbf{instance}\ \mathit{Monad}\ (\mathit{ListX}\ a)\ \textbf{where}\ \ldots$$

Then generalize from lists to arbitrary monoid.

Example: linear transformations

Assignment:

- Represent linear transformations
- Scalar, non-scalar domain & range, identity and composition

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- Represent linear transformations
- Scalar, non-scalar domain & range, identity and composition

Plan:

- Interface
- Denotation
- Representation
- Calculation (implementation)

Interface and denotation

Interface:

$$
\begin{aligned}
&\textbf{type}\ (:\!\multimap) :: * \to * \to *\\
&scale :: Num\ s \Rightarrow (s :\!\multimap s)\\
&\widehat{id} \quad :: a :\!\multimap a\\
&(\hat{\circ}) \quad :: (b :\!\multimap c) \to (a :\!\multimap b) \to (a :\!\multimap c)\\
&\ldots
\end{aligned}
$$

Interface and denotation

Interface:

$$
\begin{array}{l}
\textbf{type}\ (:\!\multimap) :: * \to * \to * \\
\\
\mathit{scale} :: \mathit{Num}\ s \Rightarrow (s :\!\multimap s) \\
\widehat{\mathit{id}} \quad :: a :\!\multimap a \\
(\hat{\circ}) \quad :: (b :\!\multimap c) \to (a :\!\multimap b) \to (a :\!\multimap c) \\
\\
\ldots
\end{array}
$$

Model:

$$
\begin{array}{l}
\textbf{type}\ a \multimap b \quad \text{-- Linear subset of } a \to b \\
\\
\mu :: (a :\!\multimap b) \to (a \multimap b)
\end{array}
$$

Interface and denotation

Interface:

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\begin{aligned}
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Model:

$$
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&\textbf{type}\ a \multimap b \quad \text{-- Linear subset of } a \to b\\
&\mu :: (a :\!\multimap b) \to (a \multimap b)
\end{aligned}
$$

Specification:

$$
\begin{aligned}
&\mu\ (scale\ s) \equiv \lambda x \to s \times x\\
&\mu\ \widehat{id} \quad\quad\ \equiv id\\
&\mu\ (g\ \hat{\circ}\ f) \quad \equiv \mu\ g \circ \mu\ f\\
&\ldots
\end{aligned}
$$

Representation

Start with 1D. Recall partial specification:

$$\mu\ (scale\ s) \equiv \lambda x \to s \times x$$

Try a direct data type representation:

$$\begin{aligned}
&\textbf{data}\ (:\!\multimap) :: * \to * \to *\ \ \textbf{where} \\
&\quad Scale :: Num\ s \Rightarrow s \to (s :\!\multimap s) \quad \text{-- ...} \\
&\mu :: (a :\!\multimap b) \to (a \multimap b) \\
&\mu\ (Scale\ s) = \lambda x \to s \times x
\end{aligned}$$

Spec trivially satisfied by $scale = Scale$.

Others are more interesting.

Calculate an implementation

Specification:

$$\mu\ \widehat{id} \equiv id$$

$$\mu\ (g \mathbin{\hat{\circ}} f) \equiv \mu\ g \circ \mu\ f$$

Calculation:

$$\begin{aligned} & id \\ \equiv\ & \lambda x \to x \\ \equiv\ & \lambda x \to 1 \times x \\ \equiv\ & \mu\ (Scale\ 1) \end{aligned}$$

$$\begin{aligned} & \mu\ (Scale\ s) \circ \mu\ (Scale\ s') \\ \equiv\ & (\lambda x \to s \times x) \circ (\lambda x' \to s' \times x') \\ \equiv\ & \lambda x' \to s \times (s' \times x') \\ \equiv\ & \lambda x' \to ((s \times s') \times x') \\ \equiv\ & \mu\ (Scale\ (s \times s')) \end{aligned}$$

Sufficient definitions:

$$\widehat{id} = Scale\ 1$$

$$Scale\ s \mathbin{\hat{\circ}} Scale\ s' = Scale\ (s \times s')$$

Algebraic abstraction

In general,

- Replace ad hoc vocabulary with a standard abstraction.
- Recast semantics as homomorphism.
- Note that laws hold.

What standard abstraction to use for $(:\multimap)$?

Category

Interface:

$$\begin{aligned}&\textbf{class } \mathit{Category}\ k \textbf{ where}\\ &\quad \mathit{id} \ :: k\ a\ a\\ &\quad (\circ) :: k\ b\ c \to k\ a\ b \to k\ a\ c\end{aligned}$$

Laws:

$$\begin{aligned}\mathit{id} \circ f &\equiv f\\ g \circ \mathit{id} &\equiv g\\ (h \circ g) \circ f &\equiv h \circ (g \circ f)\end{aligned}$$

Linear transformation category

Linear map semantics:

$$\mu :: (a :\multimap b) \to (a \multimap b)$$
$$\mu\ (Scale\ s) = \lambda x \to s \times x$$

Specification as homomorphism (no abstraction leak):

$$\mu\ id \equiv id$$
$$\mu\ (g \circ f) \equiv \mu\ g \circ \mu\ f$$

Correct-by-construction implementation:

instance *Category* $(:\multimap)$ **where**

$$id = Scale\ 1$$
$$Scale\ s \circ Scale\ s' = Scale\ (s \times s')$$

Laws for free

$$\boxed{\begin{aligned} \mu\ id &\equiv id \\ \mu\ (g \circ f) &\equiv \mu\ g \circ \mu\ f \end{aligned}} \Rightarrow \boxed{\begin{aligned} id \circ f &\equiv f \\ g \circ id &\equiv g \\ (h \circ g) \circ f &\equiv h \circ (g \circ f) \end{aligned}}$$

where equality is *semantic*.

Laws for free

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where equality is *semantic*. Proofs:

$$\boxed{\begin{aligned}&\mu\ (id \circ f)\\ \equiv\ &\mu\ id \circ \mu\ f\\ \equiv\ &id \circ \mu\ f\\ \equiv\ &\mu\ f\end{aligned}} \quad \boxed{\begin{aligned}&\mu\ (g \circ id)\\ \equiv\ &\mu\ g \circ \mu\ id\\ \equiv\ &\mu\ g \circ id\\ \equiv\ &\mu\ g\end{aligned}} \quad \boxed{\begin{aligned}&\mu\ ((h \circ g) \circ f)\\ \equiv\ &(\mu\ h \circ \mu\ g) \circ \mu\ f\\ \equiv\ &\mu\ h \circ (\mu\ g \circ \mu\ f)\\ \equiv\ &\mu\ (h \circ (g \circ f))\end{aligned}}$$

Works for other classes as well.

Higher dimensions

Interface:

$$(\vartriangle) :: (a \mathrel{:\!\!\multimap} c) \to (a \mathrel{:\!\!\multimap} d) \to (a \mathrel{:\!\!\multimap} c \times d)$$
$$(\triangledown) :: (a \mathrel{:\!\!\multimap} c) \to (b \mathrel{:\!\!\multimap} c) \to (a \times b \mathrel{:\!\!\multimap} c)$$

Semantics:

$$\mu\ (f \vartriangle g) \equiv \lambda a \to (f\ a, g\ a)$$
$$\mu\ (f \triangledown g) \equiv \lambda(a, b) \to f\ a + g\ b$$

Products and coproducts

$$\textbf{class}\ \mathit{Category}\ k \Rightarrow \mathit{ProductCat}\ k\ \textbf{where}$$

$$\quad \textbf{type}\ a \times_k b$$

$$\quad \mathit{exl} :: k\ (a \times_k b)\ a$$

$$\quad \mathit{exr} :: k\ (a \times_k b)\ b$$

$$\quad (\vartriangle) :: k\ a\ c \to k\ a\ d \to k\ a\ (c \times_k d)$$

$$\textbf{class}\ \mathit{Category}\ k \Rightarrow \mathit{CoproductCat}\ k\ \textbf{where}$$

$$\quad \textbf{type}\ a +_k b$$

$$\quad \mathit{inl} :: k\ a\ (a +_k b)$$

$$\quad \mathit{inr} :: k\ b\ (a +_k b)$$

$$\quad (\triangledown) :: k\ a\ c \to k\ b\ c \to k\ (a +_k b)\ c$$

Similar to *Arrow* and *ArrowChoice* classes.

Semantic morphisms

$$\begin{aligned}\mu\ exl &\equiv exl\\ \mu\ exr &\equiv exr\\ \mu\ (f \vartriangle g) &\equiv \mu\ f \vartriangle \mu\ g\end{aligned}$$

$$\begin{aligned}\mu\ inl &\equiv inl\\ \mu\ inr &\equiv inr\\ \mu\ (f \triangledown g) &\equiv \mu\ f \triangledown \mu\ g\end{aligned}$$

For $a \multimap b$,

$$\begin{aligned}&\textbf{type}\ a \times_{(\multimap)} b = a \times b\\ &exl\ (a, b) = a\\ &exr\ (a, b) = b\\ &f \vartriangle g = \lambda a \to (f\ a, g\ a)\end{aligned}$$

$$\begin{aligned}&\textbf{type}\ a +_{(\multimap)} b = a \times b\\ &inl\ a = (a, 0)\\ &inr\ b = (0, b)\\ &f \triangledown g = \lambda(a, b) \to f\ a + g\ b\end{aligned}$$

For calculation, see blog post *Reimagining matrices*.

Full representation and denotation

$$
\begin{aligned}
&\textbf{data}\ (:\!\multimap) :: * \to * \to * \ \ \textbf{where}\\
&\quad \mathit{Scale} :: \mathit{Num}\ s \Rightarrow s \to (s :\!\multimap s)\\
&\quad (:\!\vartriangle) :: (a :\!\multimap c) \to (a :\!\multimap d) \to (a :\!\multimap c \times d)\\
&\quad (:\!\triangledown) :: (a :\!\multimap c) \to (b :\!\multimap c) \to (a \times b :\!\multimap c)
\end{aligned}
$$

$$
\begin{aligned}
&\mu :: (a :\!\multimap b) \to (a \multimap b)\\
&\mu\ (\mathit{Scale}\ s) = \lambda x \to s \times x\\
&\mu\ (f :\!\vartriangle g) \ \ = \lambda a \to (f\ a, g\ a)\\
&\mu\ (f :\!\triangledown g) \ \ = \lambda (a, b) \to f\ a + g\ b
\end{aligned}
$$

Denotational design

Denotational design

Design methodology for typed, purely functional programming:

- Precise, simple, and compelling specification.
- Informs *use* and *implementation* without entangling.
- Standard algebraic abstractions.
- Free of abstraction leaks.
- Laws for free.
- Principled construction of correct implementation.

References

- *Denotational design with type class morphisms*
- *Push-pull functional reactive programming*
- Functional images (Pan) page with pictures & papers.
- Posts on type class morphisms
- *Reimagining matrices*
- This workshop