The simple essence of automatic differentiation

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Target

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Machine learning: promise and problems

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- AD algorithm (backprop) is complex and stateful.
- Complex graph APIs.

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- AD algorithm (backprop) is complex and stateful.
- Complex graph APIs.

Solutions:

- AD: Simple, calculated, efficient, parallel-friendly, generalized.
- API: *derivative*.

What's a derivative?

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- Number

What's a derivative?

- Number
- Vector

What's a derivative?

- Number
- Vector
- Covector

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- Matrix

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- Number
- Vector
- Covector
- Matrix
- Higher derivatives

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- Higher derivatives

Chain rule for each.

Derivatives as linear maps (Fréchet)

$$\mathcal{D} :: (a \to b) \to (a \to (a \multimap b))$$

$\mathcal{D}\ f\ a$ is a local *linear* approximation to f at a.

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$\mathcal{D}\ f\ a$ is a local *linear* approximation to f at a:

$$\lim_{\varepsilon \to 0} \frac{\|f\ (a + \varepsilon) - (f\ a + \mathcal{D}\ f\ a\ \varepsilon)\|}{\|\varepsilon\|} = 0$$

Composition

Sequential:

$$(\circ) :: (b \to c) \to (a \to b) \to (a \to c)$$
$$(g \circ f)\ a = g\ (f\ a)$$

$$\mathcal{D}\ (g \circ f)\ a = \mathcal{D}\ g\ (f\ a) \circ \mathcal{D}\ f\ a \quad \text{-- chain rule}$$

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Parallel:

$$(\vartriangle) :: (a \to c) \to (a \to d) \to (a \to c \times d)$$
$$(f \vartriangle g)\ a = (f\ a, g\ a)$$

$$\mathcal{D}\ (f \vartriangle g)\ a = \mathcal{D}\ f\ a \vartriangle \mathcal{D}\ g\ a$$

Linear functions

Linear functions

Linear functions are their own derivatives everywhere.

$$\begin{aligned}
\mathcal{D}\ id\ \ a &= id \\
\mathcal{D}\ fst\ \ a &= fst \\
\mathcal{D}\ snd\ a &= snd \\
&\ldots
\end{aligned}$$

Compositionality

Chain rule:

$$\mathcal{D}\ (g \circ f)\ a = \mathcal{D}\ g\ (f\ a) \circ \mathcal{D}\ f\ a \quad \text{-- non-compositional}$$

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To fix, combine regular result with derivative:

$$\hat{\mathcal{D}} :: (a \to b) \to (a \to (b \times (a \multimap b)))$$
$$\hat{\mathcal{D}}\ f = f \vartriangle \mathcal{D}\ f \quad \text{-- specification}$$

Often much work in common to f and $\mathcal{D}\ f$.

Abstract algebra for functions

$$\begin{aligned}&\textbf{class } \mathit{Category}\ (\rightsquigarrow) \textbf{ where}\\ &\quad \mathit{id} \ :: a \rightsquigarrow a\\ &\quad (\circ) :: (b \rightsquigarrow c) \to (a \rightsquigarrow b) \to (a \rightsquigarrow c)\end{aligned}$$

$$\begin{aligned}&\textbf{class } \mathit{Category}\ (\rightsquigarrow) \Rightarrow \mathit{Cartesian}\ (\rightsquigarrow) \textbf{ where}\\ &\quad \mathit{exl} :: (a \times b) \rightsquigarrow a\\ &\quad \mathit{exr} :: (a \times b) \rightsquigarrow b\\ &\quad (\vartriangle) :: (a \rightsquigarrow c) \to (a \rightsquigarrow d) \to (a \rightsquigarrow (c \times d))\end{aligned}$$

Plus laws and classes for arithmetic etc.

Automatic differentiation (specification)

$$\textbf{newtype}\ D\ a\ b = D\ (a \to b \times (a \multimap b))$$

$$\hat{\mathcal{D}} :: (a \to b) \to D\ a\ b$$

$$\hat{\mathcal{D}}\ f = D\ (f \vartriangle \mathcal{D}\ f) \quad \text{-- not computable}$$

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Specification: $\hat{\mathcal{D}}$ preserves *Category* and *Cartesian* structure:

$$\hat{\mathcal{D}}\ id = id$$
$$\hat{\mathcal{D}}\ (g \circ f) = \hat{\mathcal{D}}\ g \circ \hat{\mathcal{D}}\ f$$

$$\hat{\mathcal{D}}\ exl = exl$$
$$\hat{\mathcal{D}}\ exr = exr$$
$$\hat{\mathcal{D}}\ (f \vartriangle g) = \hat{\mathcal{D}}\ f \vartriangle \hat{\mathcal{D}}\ g$$

The game: solve these equations for the RHS operations.

Automatic differentiation (solution)

$$\textbf{newtype}\ D\ a\ b = D\ (a \to b \times (a \multimap b))$$

$$linearD\ f = D\ (\lambda a \to (f\ a, f))$$

$$\textbf{instance}\ Category\ D\ \textbf{where}$$
$$id = linearD\ id$$
$$D\ g \circ D\ f = D\ (\lambda a \to \textbf{let}\ \{(b, f') = f\ a; (c, g') = g\ b\}\ \textbf{in}\ (c, g' \circ f'))$$

$$\textbf{instance}\ Cartesian\ D\ \textbf{where}$$
$$exl = linearD\ exl$$
$$exr = linearD\ exr$$
$$D\ f \vartriangle D\ g = D\ (\lambda a \to \textbf{let}\ \{(b, f') = f\ a; (c, g') = g\ a\}\ \textbf{in}\ ((b, c), f' \vartriangle g'))$$

$$\textbf{instance}\ NumCat\ D\ \textbf{where}$$
$$negate = linearD\ negate$$
$$add = linearD\ add$$
$$mul = D\ (mul \vartriangle (\lambda(a, b) \to \lambda(da, db) \to b * da + a * db))$$

Running examples

$$sqr :: Num\ a \Rightarrow a \to a$$
$$sqr\ a = a * a$$

$$magSqr :: Num\ a \Rightarrow a \times a \to a$$
$$magSqr\ (a, b) = sqr\ a + sqr\ b$$

$$cosSinProd :: Floating\ a \Rightarrow a \times a \to a \times a$$
$$cosSinProd\ (x, y) = (cos\ z, sin\ z)\ \textbf{where}\ z = x * y$$

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In categorical vocabulary:

$$sqr = mul \circ (id \vartriangle id)$$

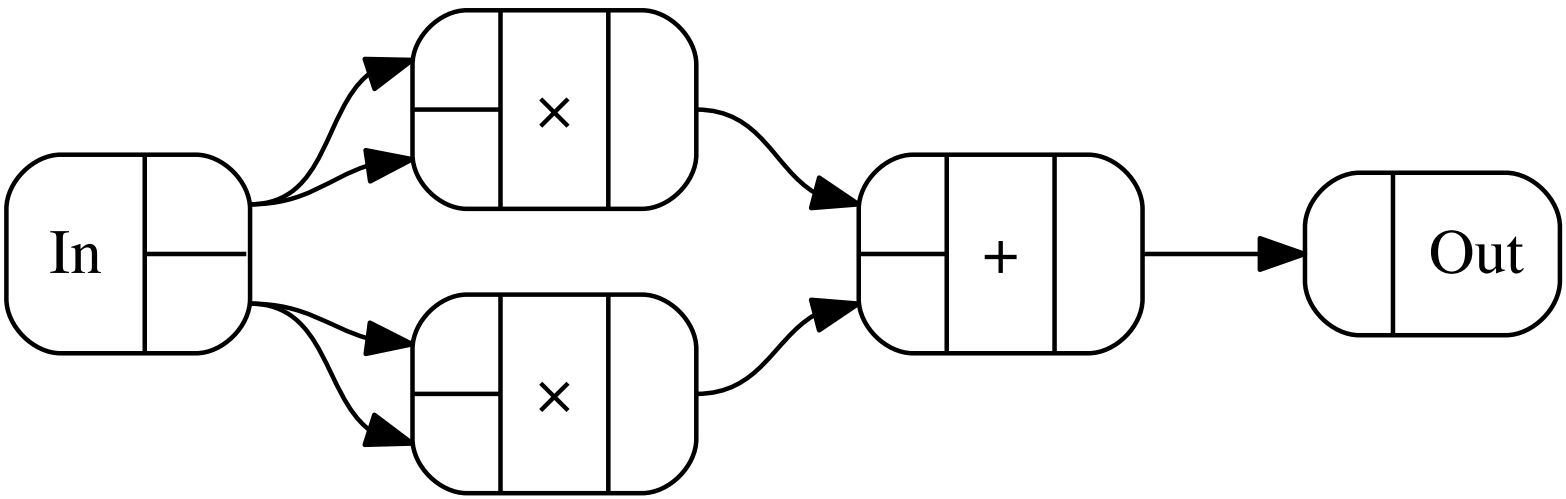
$$magSqr = add \circ ((sqr \circ exl) \vartriangle (sqr \circ exr))$$

$$cosSinProd = (cos \vartriangle sin) \circ mul$$

Visualizing computations

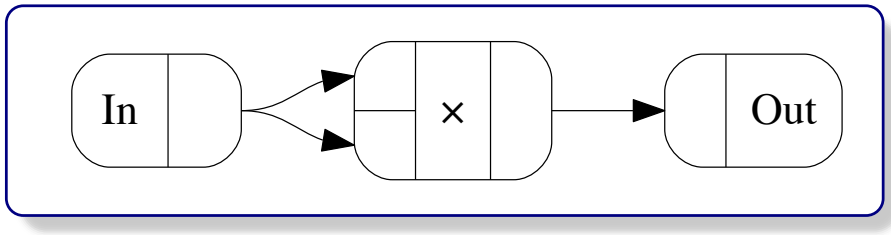
$$magSqr\ (a, b) = sqr\ a + sqr\ b$$

$$magSqr = add \circ ((sqr \circ exl) \vartriangle (sqr \circ exr))$$



Auto-generated from Haskell code. See *Compiling to categories*.

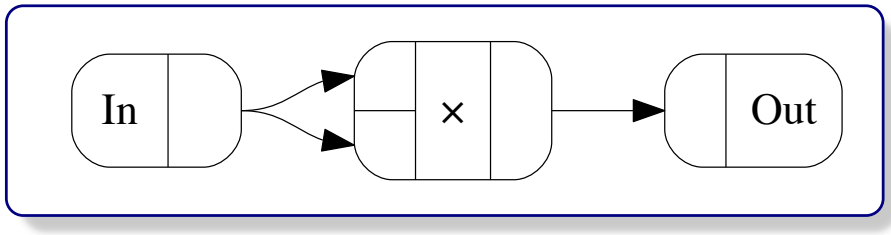
AD example



$$sqr\ a = a * a$$

$$sqr = mul \circ (id \vartriangle id)$$

AD example



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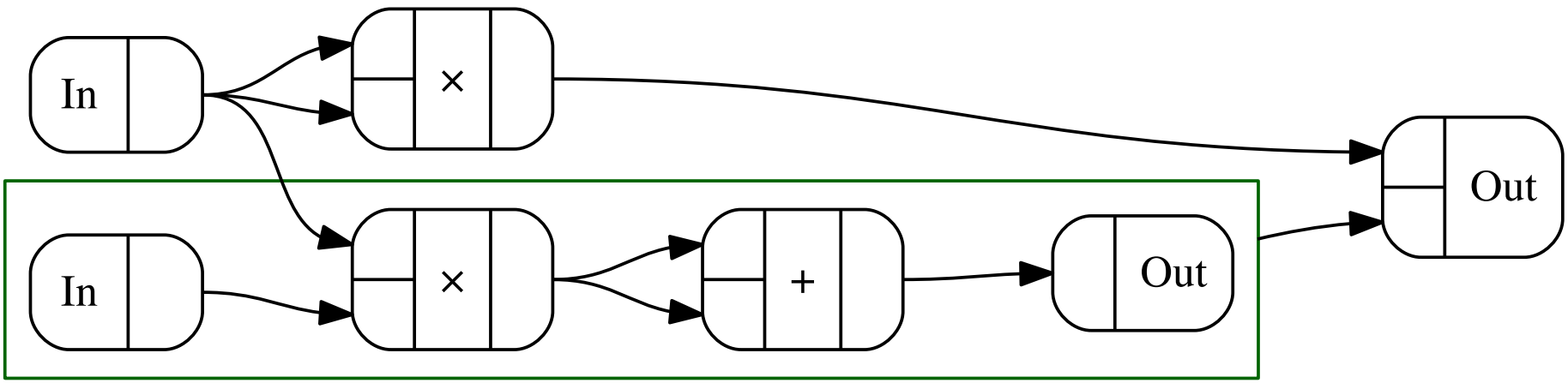

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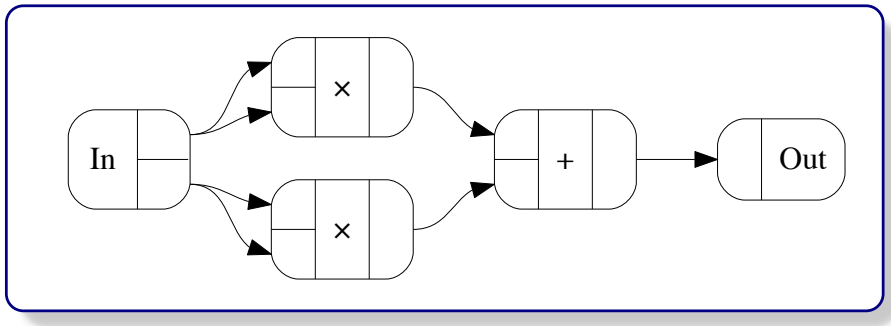

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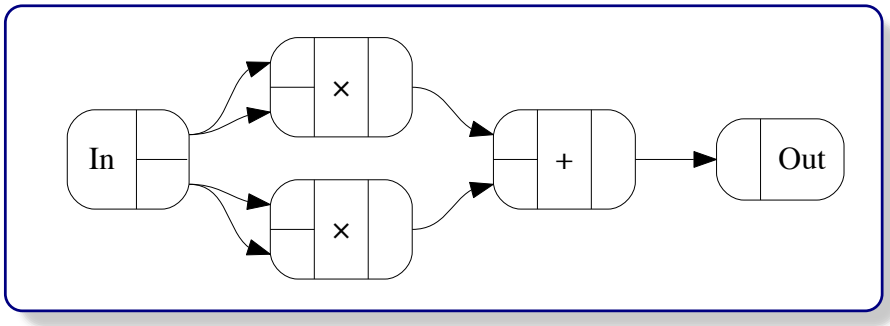
$$magSqr = add \circ ((sqr \circ exl) \vartriangle (sqr \circ exr))$$

AD example



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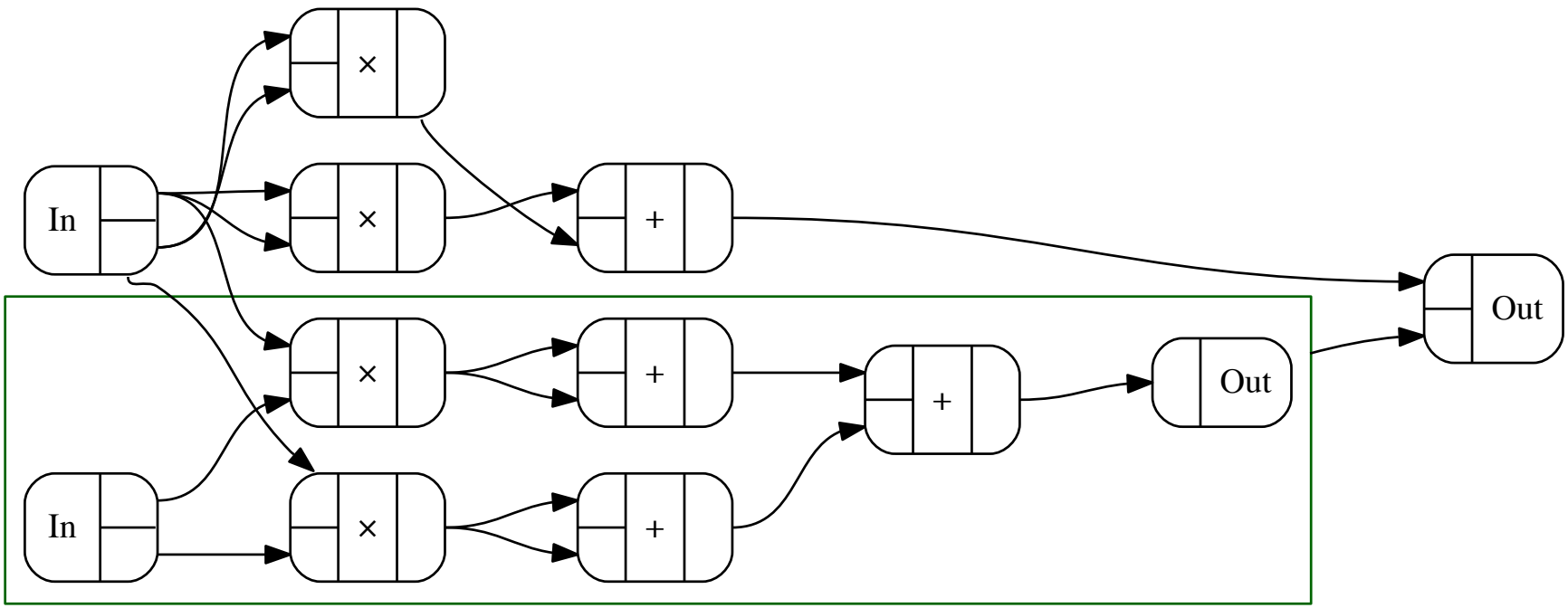

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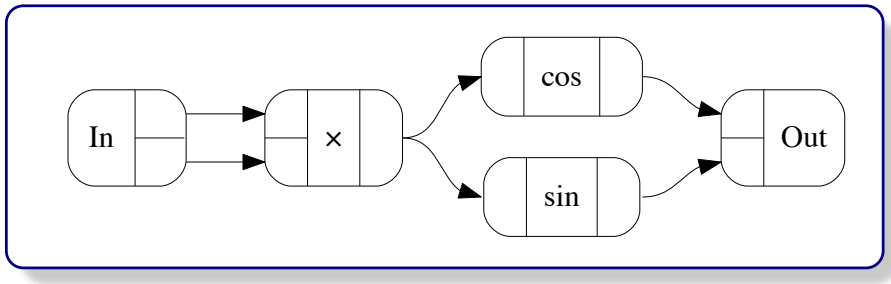
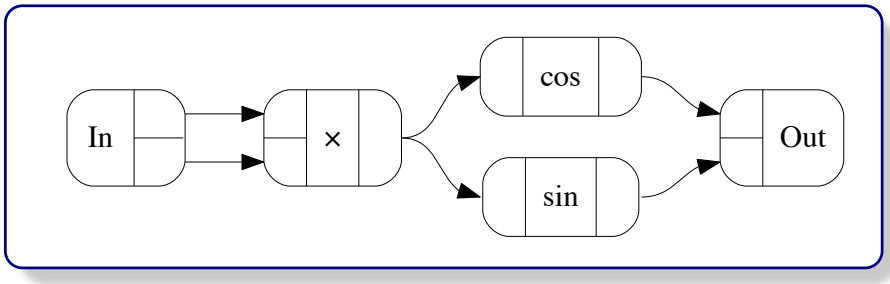


$$cosSinProd\ (x, y) = (cos\ z, sin\ z)\ \textbf{where}\ z = x * y$$

$$cosSinProd = (cos \vartriangle sin) \circ mul$$

AD example



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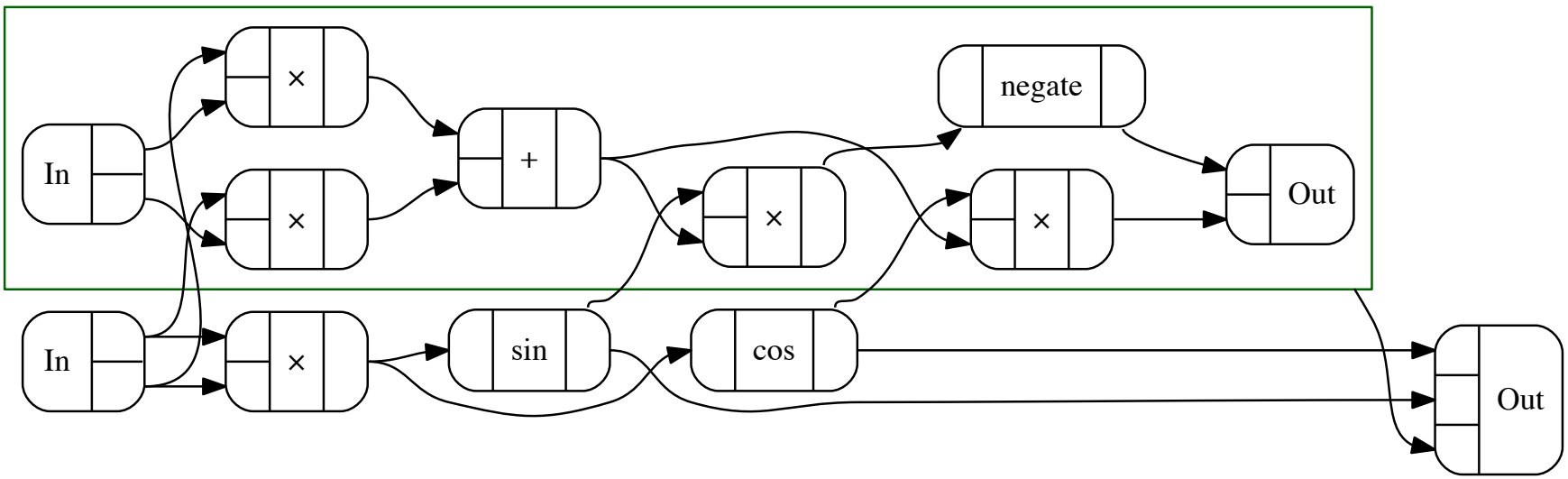

Generalizing AD

$$\textbf{newtype}\ D\ a\ b = D\ (a \to b \times (a \multimap b))$$

$$linearD\ f = D\ (\lambda a \to (f\ a, f))$$

$$\textbf{instance}\ Category\ D\ \textbf{where}$$
$$id = linearD\ id$$
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$$\textbf{instance}\ Cartesian\ D\ \textbf{where}$$
$$exl = linearD\ exl$$
$$exr = linearD\ exr$$
$$D\ f \mathbin{\triangle} D\ g = D\ (\lambda a \to \textbf{let}\ \{(b, f') = f\ a; (c, g') = g\ a\}\ \textbf{in}\ ((b, c), f' \mathbin{\triangle} g'))$$

Each D operation just uses corresponding $(\multimap)$ operation.

Generalize from $(\multimap)$ to other cartesian categories.

Generalized AD

$$\textbf{newtype}\ D_{(\rightsquigarrow)}\ a\ b = D\ (a \to b \times (a \rightsquigarrow b))$$

$$linearD\ f\ f' = D\ (\lambda a \to (f\ a, f'))$$

$$\textbf{instance}\ Category\ (\rightsquigarrow) \Rightarrow Category\ D_{(\rightsquigarrow)}\ \textbf{where}$$

$$id = linearD\ id\ id$$

$$D\ g \circ D\ f = D\ (\lambda a \to \textbf{let}\ \{(b, f') = f\ a; (c, g') = g\ b\}\ \textbf{in}\ (c, g' \circ f'))$$

$$\textbf{instance}\ Cartesian\ (\rightsquigarrow) \Rightarrow Cartesian\ D_{(\rightsquigarrow)}\ \textbf{where}$$

$$exl = linearD\ exl\ exl$$

$$exr = linearD\ exr\ exr$$

$$D\ f \vartriangle D\ g = D\ (\lambda a \to \textbf{let}\ \{(b, f') = f\ a; (c, g') = g\ a\}\ \textbf{in}\ ((b, c), f' \vartriangle g'))$$

$$\textbf{instance}\ \ldots \Rightarrow NumCat\ D\ \textbf{where}$$

$$negate = linearD\ negate\ negate$$

$$add = linearD\ add\ add$$

$$mul = ??$$

Numeric operations

Specific to (linear) *functions*:

$$mul = D\ (mul \vartriangle (\lambda(a, b) \to \lambda(da, db) \to b * da + a * db))$$

Numeric operations

Specific to (linear) *functions*:

$$mul = D\ (mul \vartriangle (\lambda(a, b) \to \lambda(da, db) \to b * da + a * db))$$

Rephrase:

$$scale :: Multiplicative\ a \Rightarrow a \to (a \to a)$$
$$scale\ u = \lambda v \to u * v$$

$$(\triangledown) :: (a \to c) \to (b \to c) \to ((a \times b) \to c)$$
$$(f \triangledown g)\ (a, b) = f\ a + g\ b$$

Now

$$mul = D\ (mul \vartriangle (\lambda(a, b) \to scale\ b \triangledown scale\ a))$$

New generalized vocabulary

$$\begin{aligned}&\textbf{class } \mathit{Category}\ (\rightsquigarrow) \Rightarrow \mathit{Cocartesian}\ (\rightsquigarrow) \textbf{ where}\\ &\quad \mathit{inl} :: a \rightsquigarrow (a \times b)\\ &\quad \mathit{inr} :: b \rightsquigarrow (a \times b)\\ &\quad (\triangledown) :: (a \rightsquigarrow c) \to (b \rightsquigarrow c) \to ((a \times b) \rightsquigarrow c)\end{aligned}$$

$$\begin{aligned}&\textbf{class } \mathit{ScalarCat}\ (\rightsquigarrow)\ a \textbf{ where}\\ &\quad \mathit{scale} :: a \to (a \rightsquigarrow a)\end{aligned}$$

Differentiation:

$$\mathcal{D}\ (f \triangledown g)\ (a, b) = \mathcal{D}\ f\ \ a \triangledown \mathcal{D}\ g\ \ b$$

The rest are linear.

Linear transformations as functions

$$\textbf{newtype}\ a \mathbin{\overset{+}{\to}} b = \mathit{AddFun}\ (a \to b)$$

$$\begin{aligned}
&\textbf{instance}\ \mathit{Category}\ (\overset{+}{\to})\ \textbf{where}\\
&\quad \mathit{id}\ \ = \mathit{AddFun}\ \mathit{id}\\
&\quad (\circ) = \mathit{inNew}_2\ (\circ)
\end{aligned}$$

$$\begin{aligned}
&\textbf{instance}\ \mathit{Cartesian}\ (\overset{+}{\to})\ \textbf{where}\\
&\quad \mathit{exl}\ = \mathit{AddFun}\ \mathit{exl}\\
&\quad \mathit{exr} = \mathit{AddFun}\ \mathit{exr}\\
&\quad (\triangle)\ = \mathit{inNew}_2\ (\triangle)
\end{aligned}$$

$$\begin{aligned}
&\textbf{instance}\ \mathit{Cocartesian}\ (\overset{+}{\to})\ \textbf{where}\\
&\quad \mathit{inl}\ = \mathit{AddFun}\ (\lambda a \to (a, 0))\\
&\quad \mathit{inr}\ = \mathit{AddFun}\ (\lambda b \to (0, b))\\
&\quad (\triangledown) = \mathit{inNew}_2\ (\lambda f\ g\ (a, b) \to f\ a + g\ b)
\end{aligned}$$

$$\begin{aligned}
&\textbf{instance}\ \mathit{Multiplicative}\ s \Rightarrow \mathit{ScalarCat}\ (\overset{+}{\to})\ s\ \textbf{where}\\
&\quad \mathit{scale}\ s = \mathit{AddFun}\ (s *)
\end{aligned}$$

Extracting a data representation

- How to extract a matrix or gradient vector?
- Sample over a domain *basis* (rows of identity matrix).
- For n-dimensional *domain*,
  - Make n passes.
  - Each pass works on n-D sparse ("one-hot") input.
  - Very inefficient.
- For gradient-based optimization,
  - High-dimensional domain.
  - Very low-dimensional (1-D) codomain.

Generalized matrices

$$\textbf{newtype}\ M_s\ a\ b = \ldots$$

$$applyM :: M_s\ a\ b \to (a \to b)$$

Require *applyM* to preserve structure. Solve for methods.

Compositional matrices

Sufficient to build arbitrary "matrices":

$$
\begin{array}{lll}
\mathit{scale} :: a \to (a \rightsquigarrow a) & & \text{-- } 1 \times 1 \\
(\triangledown) :: (a \rightsquigarrow c) \to (b \rightsquigarrow c) \to ((a \times b) \rightsquigarrow c) & & \text{-- horizontal juxt} \\
(\triangle) :: (a \rightsquigarrow c) \to (a \rightsquigarrow d) \to (a \rightsquigarrow (c \times d)) & & \text{-- vertical juxt}
\end{array}
$$

Types guarantee rectangularity.

Efficiency of composition

- Composition is associative.
- Some associations are more efficient than others, so
  - Associate optimally.
  - Equivalent to *matrix chain multiplication* — $O(n \log n)$.
  - Choice determined by *types*, i.e., compile-time information.

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- Composition is associative.
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  - Associate optimally.
  - Equivalent to *matrix chain multiplication* — $O(n \log n)$.
  - Choice determined by *types*, i.e., compile-time information.
- All right: "forward mode AD" (FAD).
- All left: "reverse mode AD" (RAD).
- RAD is *much* better for gradient-based optimization.

Left-associating composition (RAD)

CPS-like category:

- Represent $a \rightsquigarrow b$ by $(b \rightsquigarrow r) \to (a \rightsquigarrow r)$.
- Meaning: $f' \mapsto (\lambda h \to h \circ f')$.
- Construct $h \circ \mathcal{D}\ f\ a$ directly, without $\mathcal{D}\ f\ a$.

Old technique (Cayley 1854), vastly generalized by Yoneda.

Continuation category (specification)

$$\textbf{newtype}\ Cont^{r}_{(\rightsquigarrow)}\ a\ b = Cont\ ((b \rightsquigarrow r) \to (a \rightsquigarrow r))$$

$$cont :: Category\ (\rightsquigarrow) \Rightarrow (a \rightsquigarrow b) \to Cont^{r}_{(\rightsquigarrow)}\ a\ b$$
$$cont\ f = Cont\ (\circ f)$$

Require *cont* to preserve structure. Solve for methods.

Continuation category (specification)

$$\textbf{newtype}\ Cont^{r}_{(\rightsquigarrow)}\ a\ b = Cont\ ((b \rightsquigarrow r) \to (a \rightsquigarrow r))$$

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$$cont\ f = Cont\ (\circ f)$$

Require *cont* to preserve structure. Solve for methods.

We'll use an isomorphism:

$$join :: Cocartesian\ (\rightsquigarrow) \Rightarrow (c \rightsquigarrow a) \times (d \rightsquigarrow a) \to ((c \times d) \rightsquigarrow a)$$
$$unjoin :: Cocartesian\ (\rightsquigarrow) \Rightarrow ((c \times d) \rightsquigarrow a) \to (c \rightsquigarrow a) \times (d \rightsquigarrow a)$$

$$join\ (f, g) = f \triangledown g$$
$$unjoin\ h = (h \circ inl, h \circ inr)$$

Continuation category (solution)

$$\textbf{instance } \mathit{Category}\ (\rightsquigarrow) \Rightarrow \mathit{Category}\ \mathit{Cont}^r_{(\rightsquigarrow)} \textbf{ where}$$

$$\begin{aligned} \mathit{id} &= \mathit{Cont}\ \mathit{id} \\ \mathit{Cont}\ g \circ \mathit{Cont}\ f &= \mathit{Cont}\ (f \circ g) \end{aligned}$$

$$\textbf{instance } \mathit{Cartesian}\ (\rightsquigarrow) \Rightarrow \mathit{Cartesian}\ \mathit{Cont}^r_{(\rightsquigarrow)} \textbf{ where}$$

$$\begin{aligned} \mathit{exl} &= \mathit{Cont}\ (\mathit{join} \circ \mathit{inl}) \\ \mathit{exr} &= \mathit{Cont}\ (\mathit{join} \circ \mathit{inr}) \\ (\vartriangle) &= \mathit{inNew}_2\ (\lambda f\ g \rightarrow (f \vartriangledown g) \circ \mathit{unjoin}) \end{aligned}$$

$$\textbf{instance } \mathit{Cocartesian}\ (\rightsquigarrow) \Rightarrow \mathit{Cocartesian}\ \mathit{Cont}^r_{(\rightsquigarrow)} \textbf{ where}$$

$$\begin{aligned} \mathit{inl} &= \mathit{Cont}\ (\mathit{exl} \circ \mathit{unjoin}) \\ \mathit{inr} &= \mathit{Cont}\ (\mathit{exr} \circ \mathit{unjoin}) \\ (\vartriangledown) &= \mathit{inNew}_2\ (\lambda f\ g \rightarrow \mathit{join} \circ (f \vartriangle g)) \end{aligned}$$

$$\textbf{instance } \mathit{ScalarCat}\ (\rightsquigarrow)\ a \Rightarrow \mathit{ScalarCat}\ \mathit{Cont}^r_{(\rightsquigarrow)}\ a \textbf{ where}$$

$$\mathit{scale}\ s = \mathit{Cont}\ (\mathit{scale}\ s)$$

Reverse-mode AD without tears

Reverse-mode AD without tears

$$D_{Cont^r_{M_s}}$$

Left-associating composition (RAD)

- CPS-like category:
  - Represent $a \rightsquigarrow b$ by $(b \rightsquigarrow r) \to (a \rightsquigarrow r)$.
  - Meaning: $f \mapsto (\circ f)$.
  - Results in left-composition.
  - Initialize with $id :: r \rightsquigarrow r$.
  - Construct $h \circ \mathcal{D}\ f\ a$ directly, without $\mathcal{D}\ f\ a$.

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  - Initialize with $id :: r \rightsquigarrow r$.
  - Construct $h \circ \mathcal{D}\ f\ a$ directly, without $\mathcal{D}\ f\ a$.
- We've seen this trick before:
  - Transforming naive *reverse* from quadratic to linear.
  - List generalizes to monoids, and monoids to categories.

One of my favorite papers

- *Continuation-Based Program Transformation Strategies*
  Mitch Wand, 1980, JACM.

- Introduce a continuation argument, e.g., $[a] \to [a]$.

- Notice the continuations that arise, e.g., $(+\!\!+\ as)$.

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- Introduce a continuation argument, e.g., $[a] \to [a]$.

- Notice the continuations that arise, e.g., $(+\!\!+\ as)$.

- Find a *data* representation, e.g., $as :: [a]$

- Identify associative operation that represents composition, e.g., $(+\!\!+)$, since $(+\!\!+\ bs) \circ (+\!\!+\ as) = (+\!\!+\ (as +\!\!+ bs))$.

Duality

- Vector space dual: $u \multimap s$, with u a vector space over s.
- If u has finite dimension, then $u \multimap s \cong u$.
- For $f :: u \multimap s$, $f = dot\ v$ for some $v :: u$.
- Gradients are derivatives of functions with scalar codomain.
- Represent $a \multimap b$ by $(b \multimap s) \to (a \multimap s)$ by $b \to a$.
- *Ideal* for extracting gradient vector. Just apply to 1 (id).

Duality (specification)

$$\textbf{newtype}\ Dual_{(\rightsquigarrow)}\ a\ b = Dual\ (b \rightsquigarrow a)$$

$$asDual :: Cont^{s}_{(\rightsquigarrow)}\ a\ b \to Dual_{(\rightsquigarrow)}\ a\ b$$
$$asDual\ (Cont\ f) = Dual\ (dot^{-1} \circ f \circ dot)$$

where

$$dot \quad :: u \to (u \multimap s)$$
$$dot^{-1} :: (u \multimap s) \to u$$

Require *asDual* to preserve structure. Solve for methods.

Duality (solution)

$$\textbf{newtype}\ \mathit{Dual}_{(\rightsquigarrow)}\ a\ b = \mathit{Dual}\ (b \rightsquigarrow a)$$

$$\textbf{instance}\ \mathit{Category}\ (\rightsquigarrow) \Rightarrow \mathit{Category}\ \mathit{Dual}_{(\rightsquigarrow)}\ \textbf{where}$$
$$\begin{aligned} \mathit{id} &= \mathit{Dual}\ \mathit{id} \\ (\circ) &= \mathit{inNew}_2\ (\mathit{flip}\ (\circ)) \end{aligned}$$

$$\textbf{instance}\ \mathit{Cocartesian}\ (\rightsquigarrow) \Rightarrow \mathit{Cartesian}\ \mathit{Dual}_{(\rightsquigarrow)}\ \textbf{where}$$
$$\begin{aligned} \mathit{exl} &= \mathit{Dual}\ \mathit{inl} \\ \mathit{exr} &= \mathit{Dual}\ \mathit{inr} \\ (\triangle) &= \mathit{inNew}_2\ (\triangledown) \end{aligned}$$

$$\textbf{instance}\ \mathit{Cartesian}\ (\rightsquigarrow) \Rightarrow \mathit{Cocartesian}\ \mathit{Dual}_{(\rightsquigarrow)}\ \textbf{where}$$
$$\begin{aligned} \mathit{inl} &= \mathit{Dual}\ \mathit{exl} \\ \mathit{inr} &= \mathit{Dual}\ \mathit{exr} \\ (\triangledown) &= \mathit{inNew}_2\ (\triangle) \end{aligned}$$

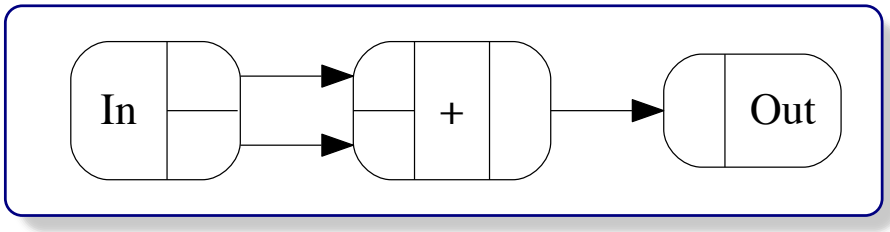
$$\textbf{instance}\ \mathit{ScalarCat}\ (\rightsquigarrow)\ s \Rightarrow \mathit{ScalarCat}\ \mathit{Dual}_{(\rightsquigarrow)}\ s\ \textbf{where}$$
$$\mathit{scale}\ s = \mathit{Dual}\ (\mathit{scale}\ s)$$

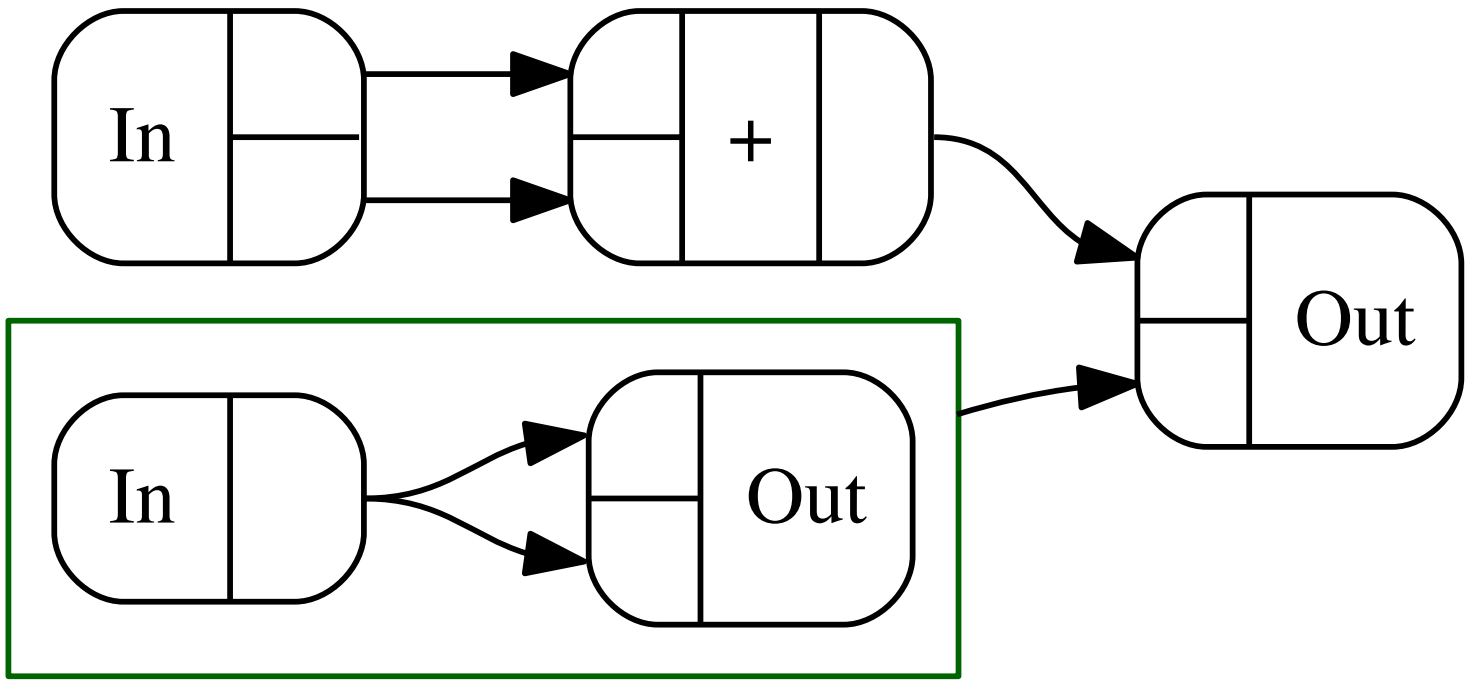
Backpropagation

Backpropagation

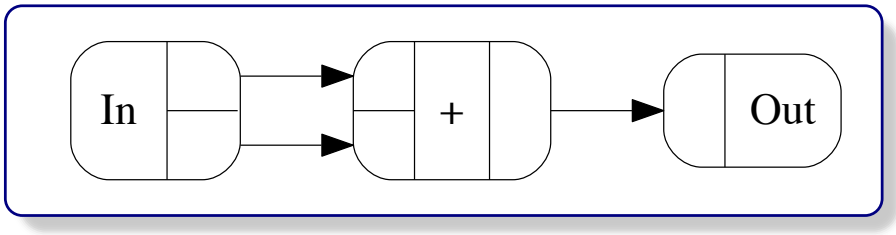
$$D_{Dual_{\rightarrow}+}$$

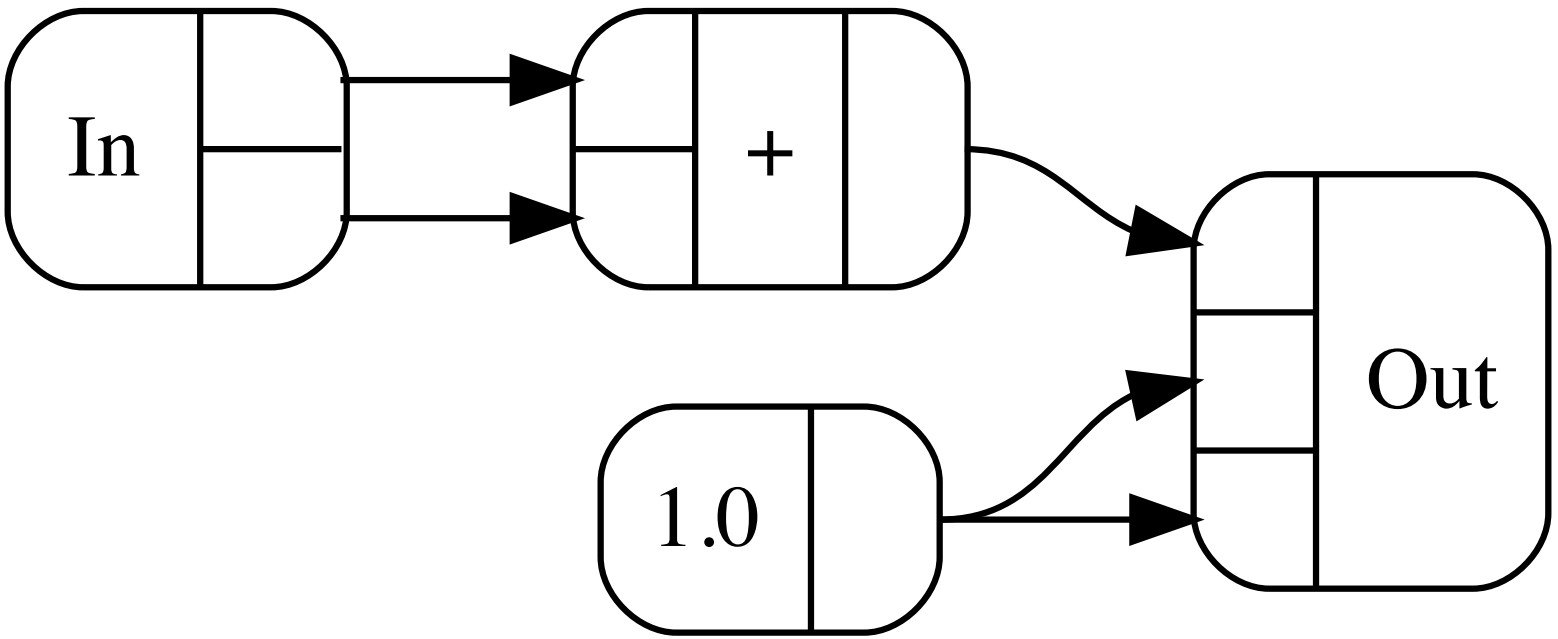
RAD example (dual function)





RAD example (dual vector)





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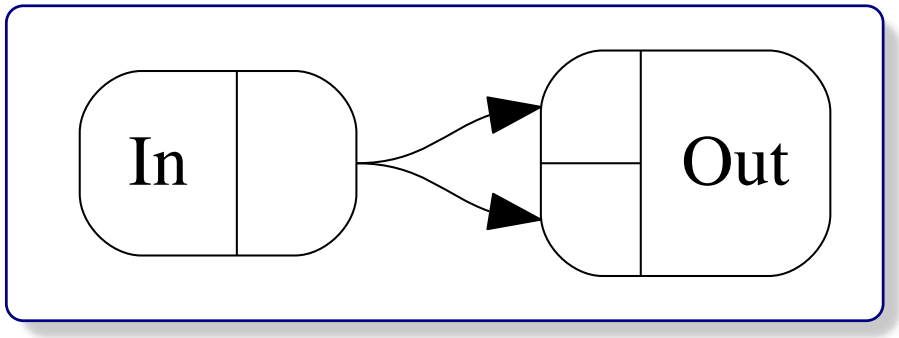
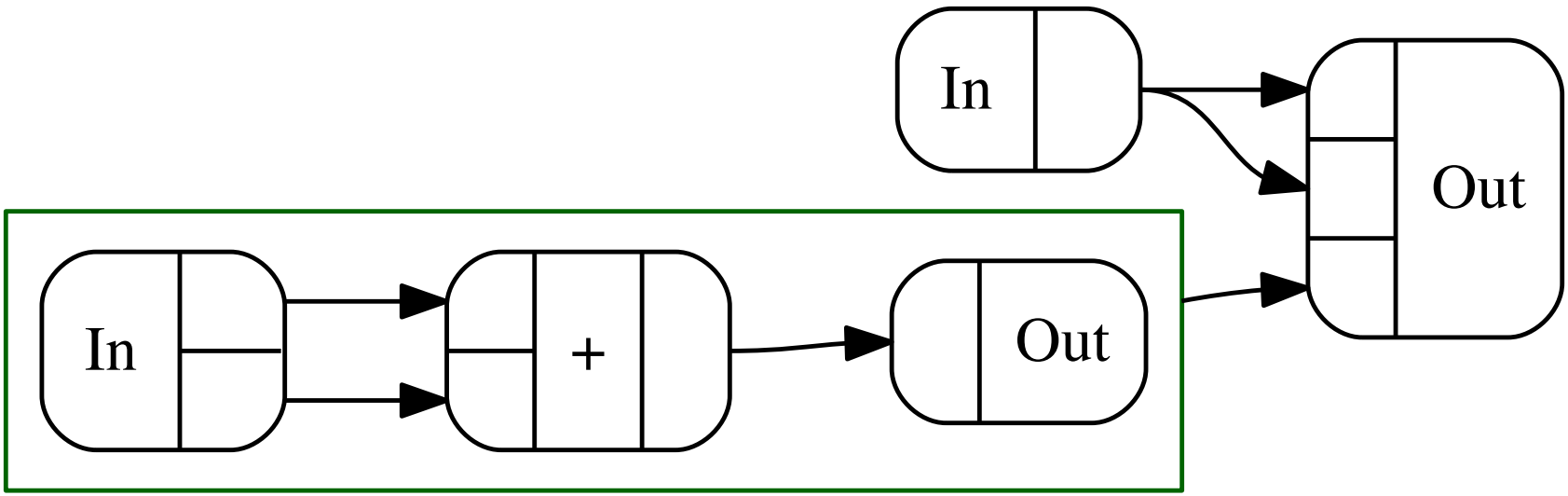
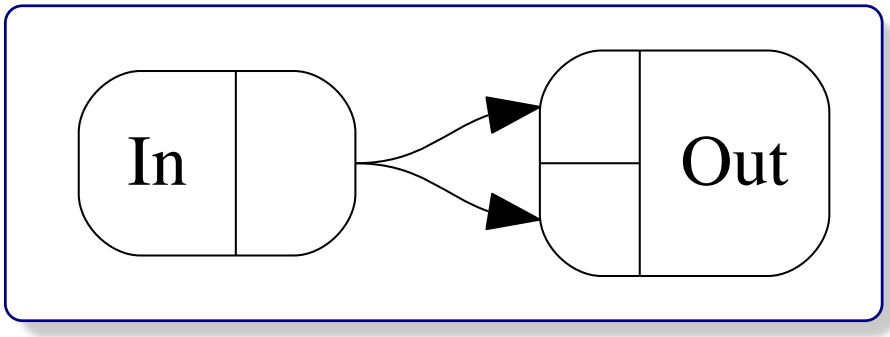
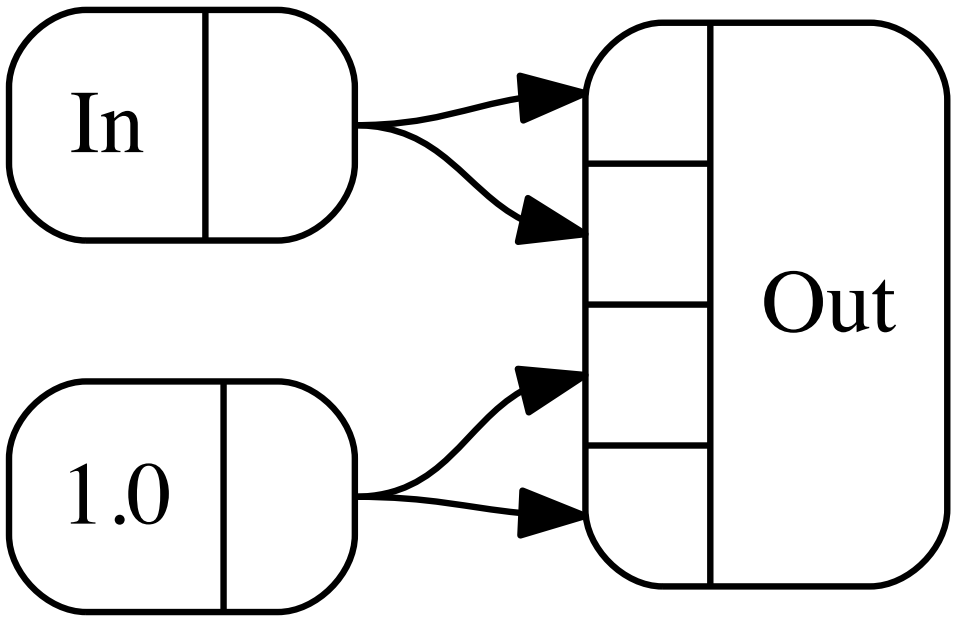


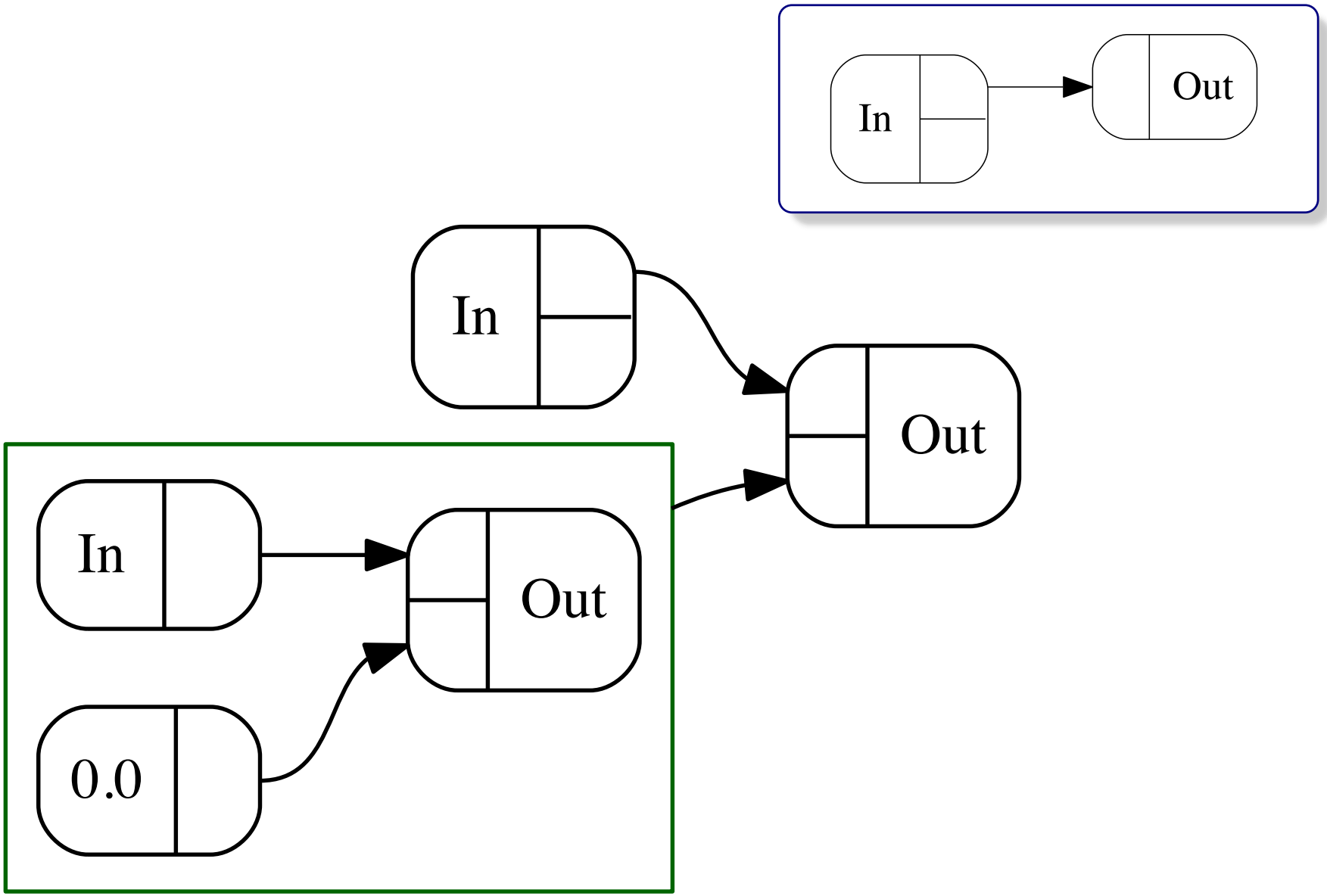


RAD example (vector)

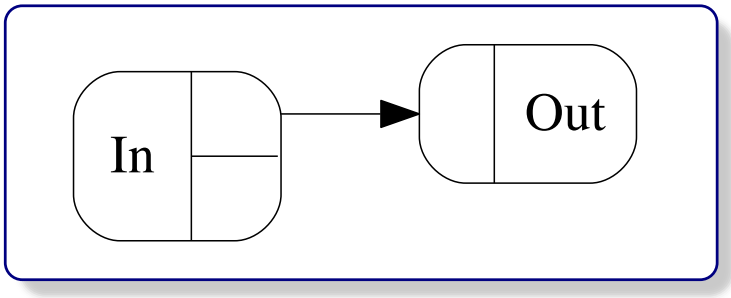


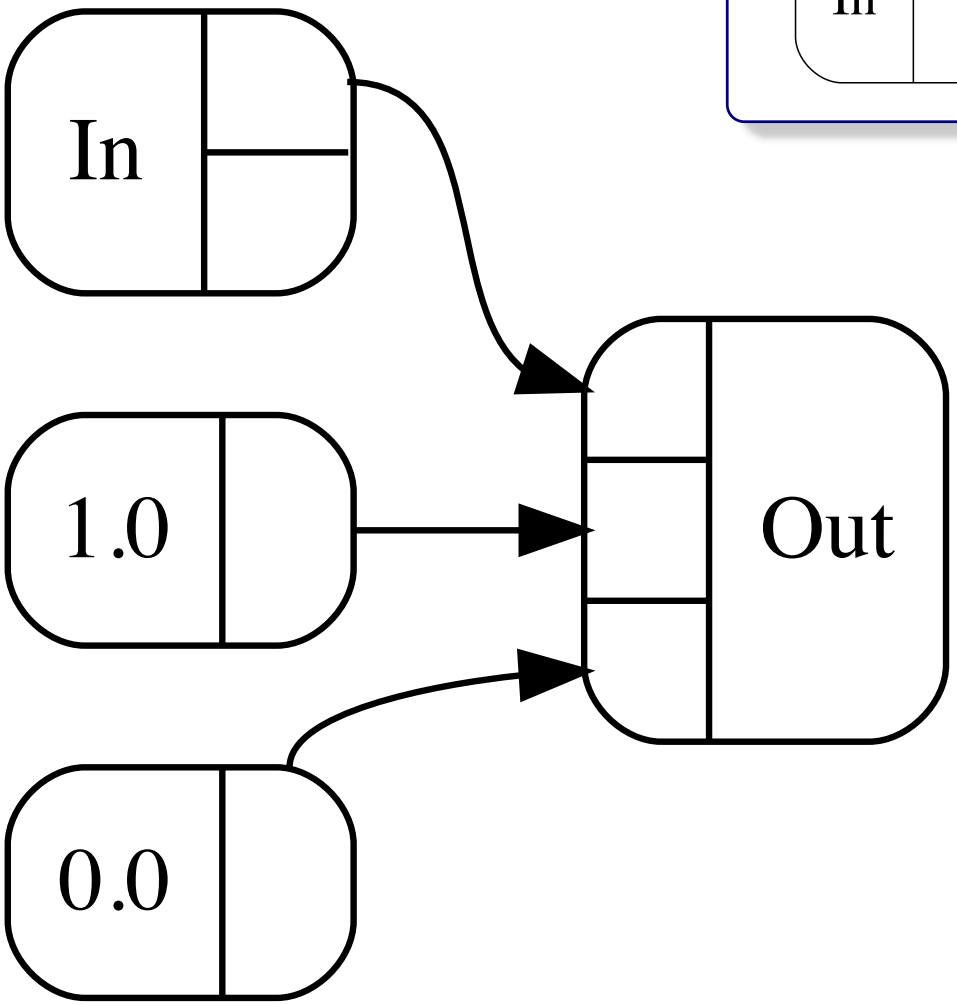


RAD example (dual function)

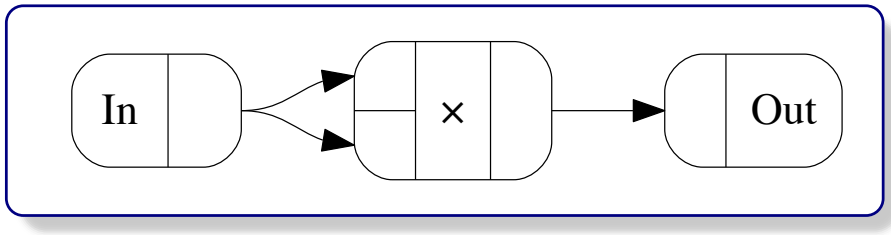


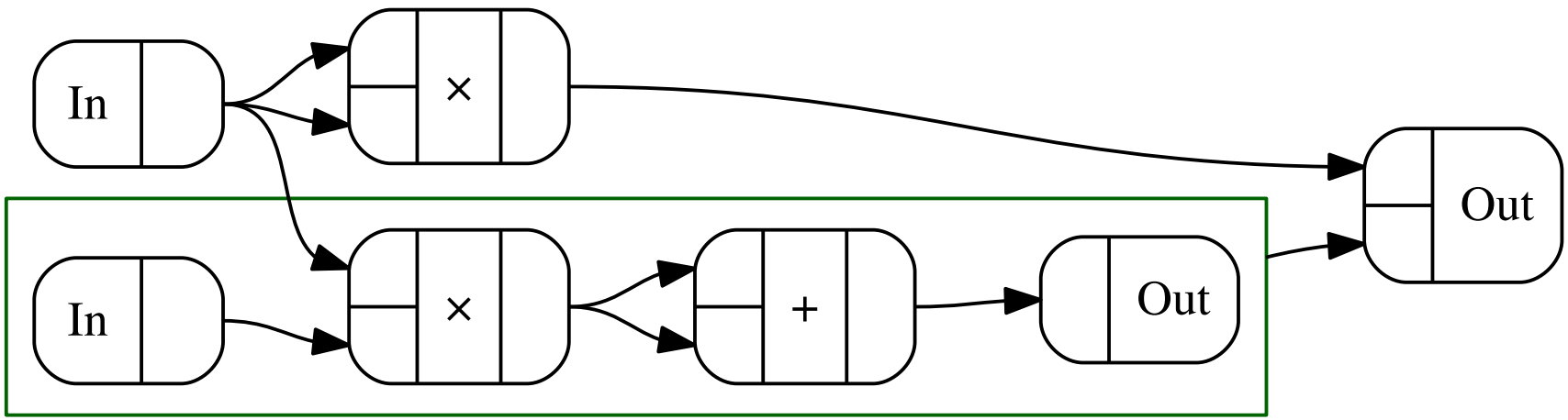
RAD example (dual vector)





RAD example (dual function)





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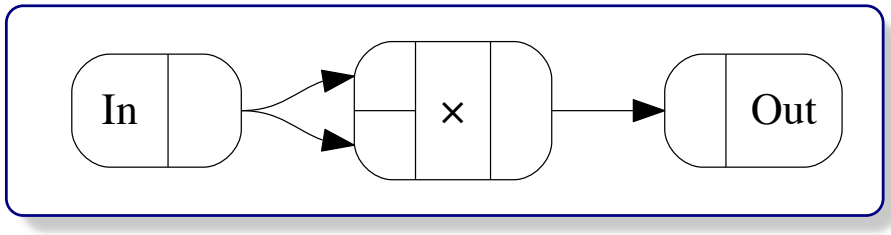

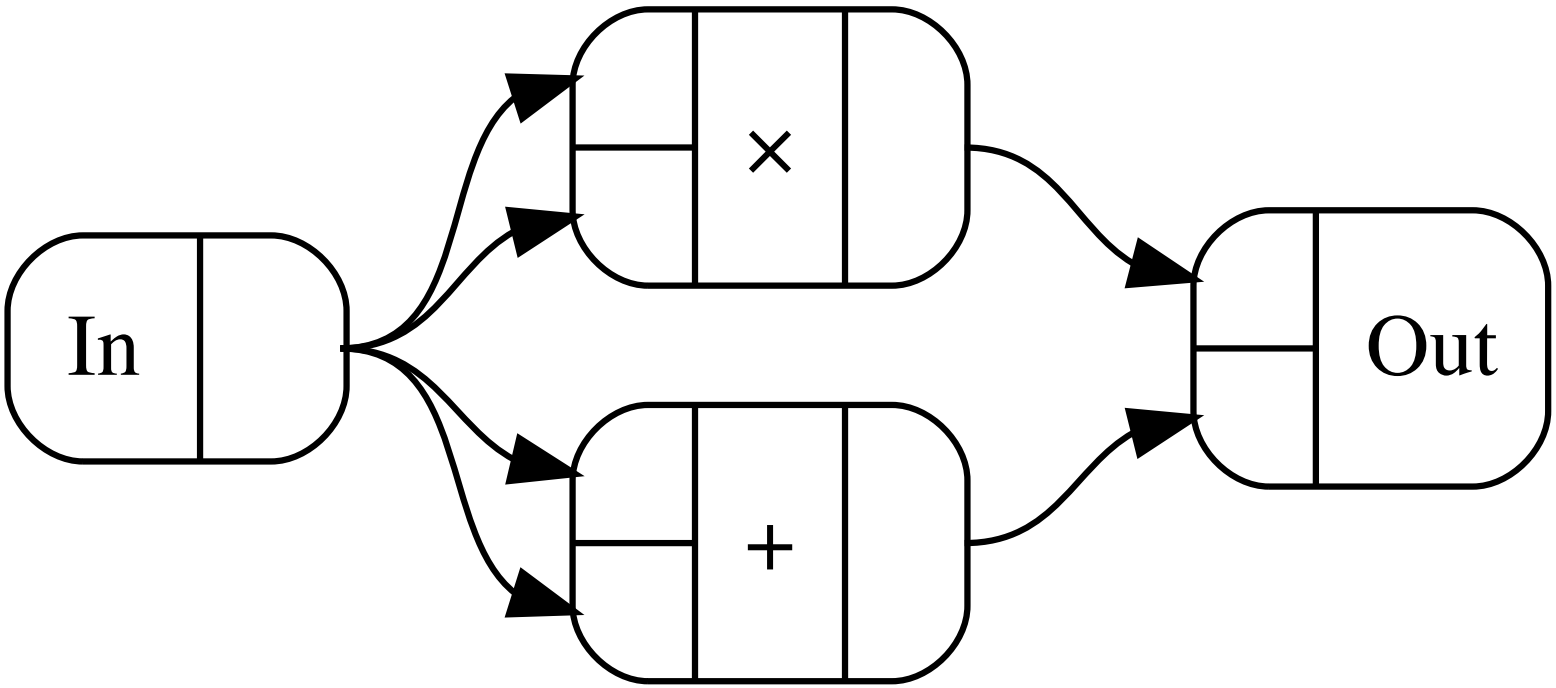

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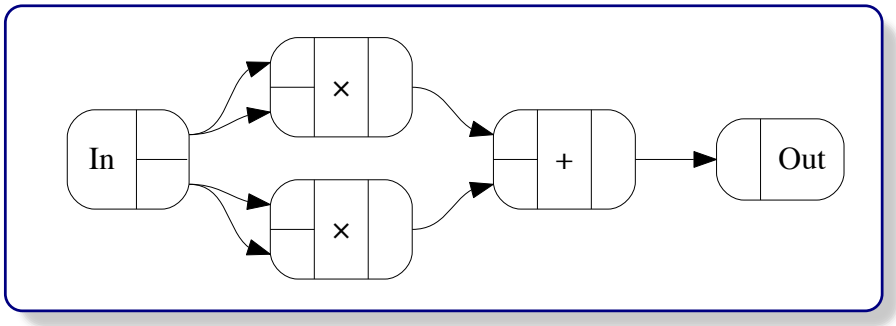

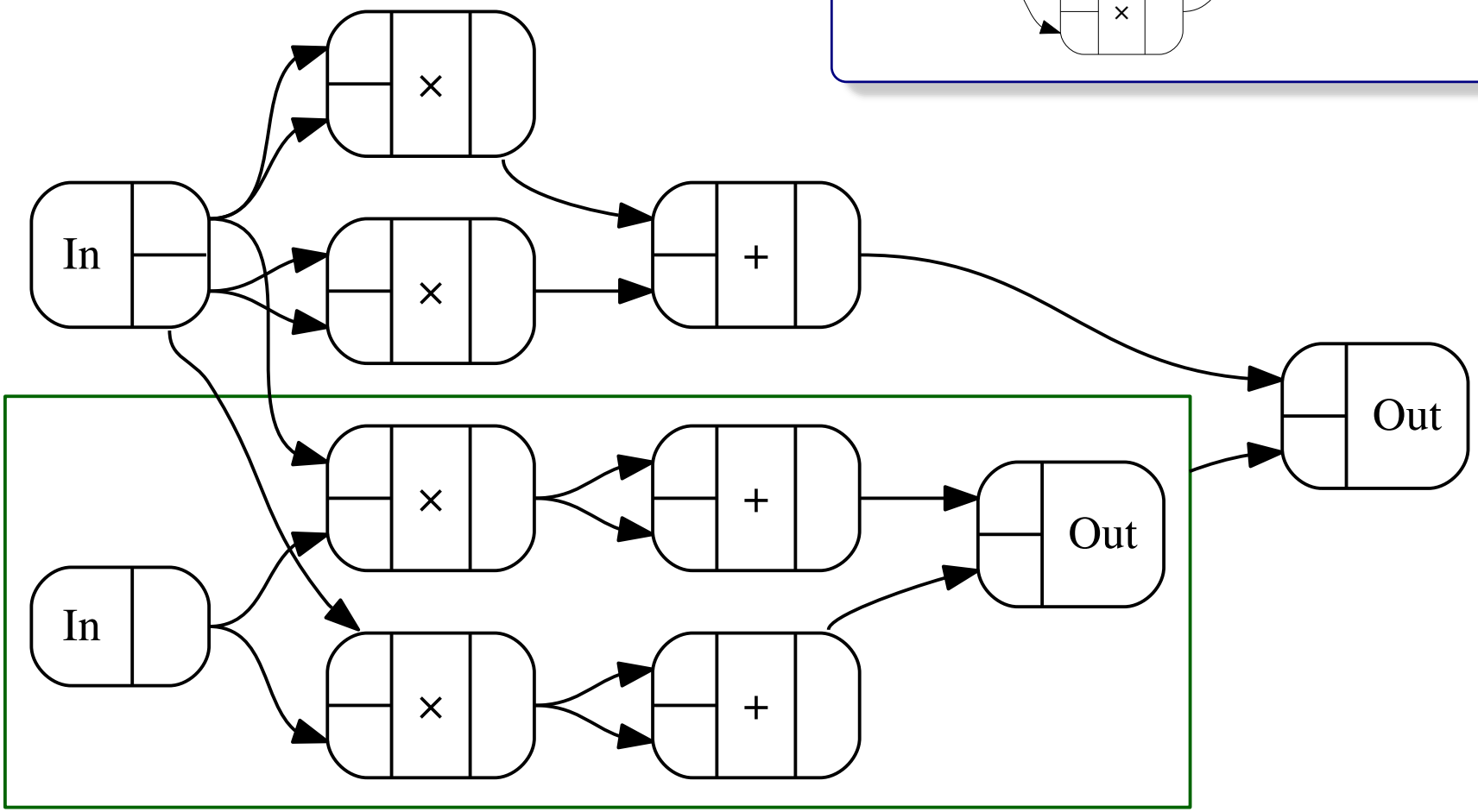

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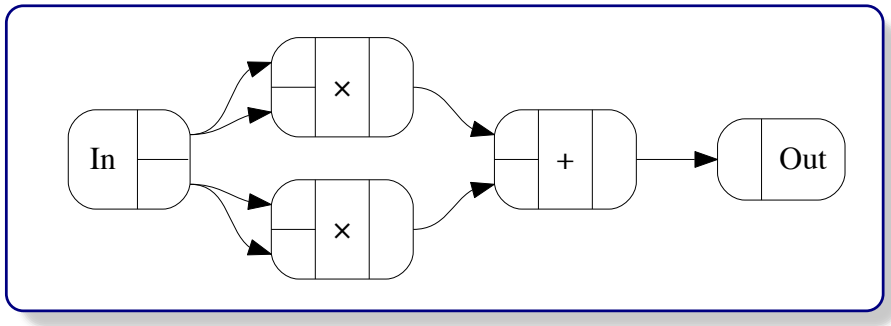
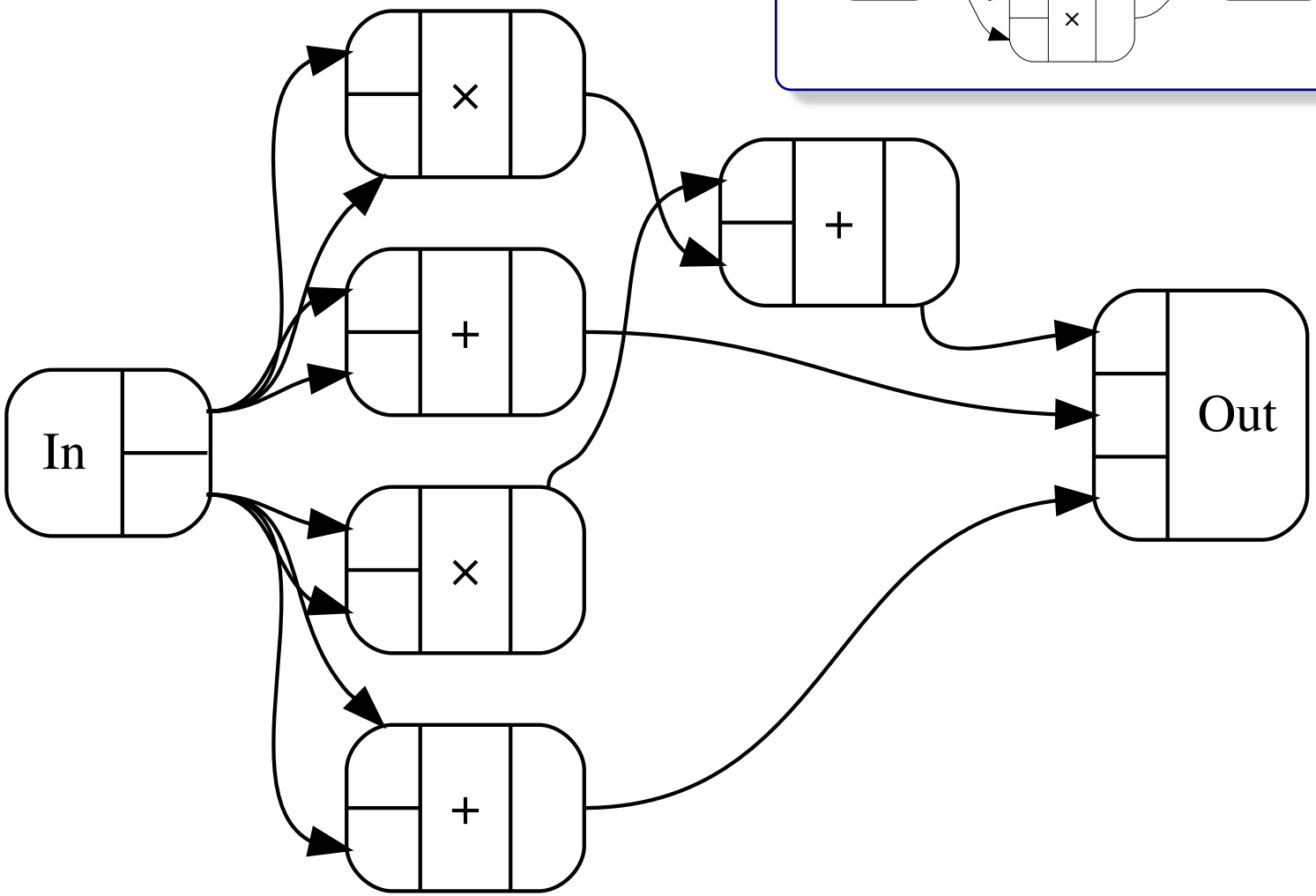
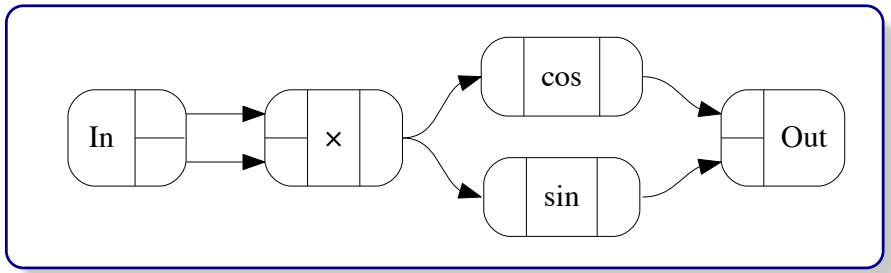
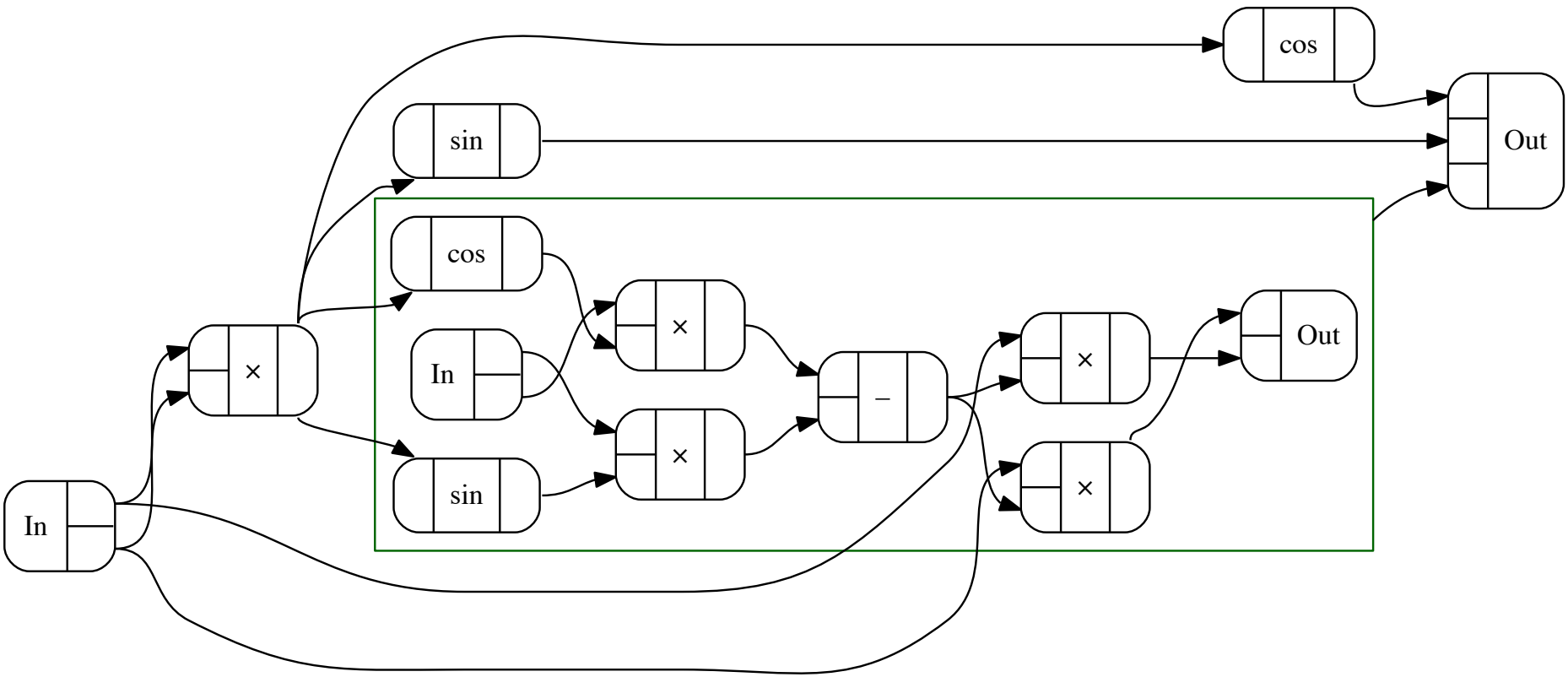


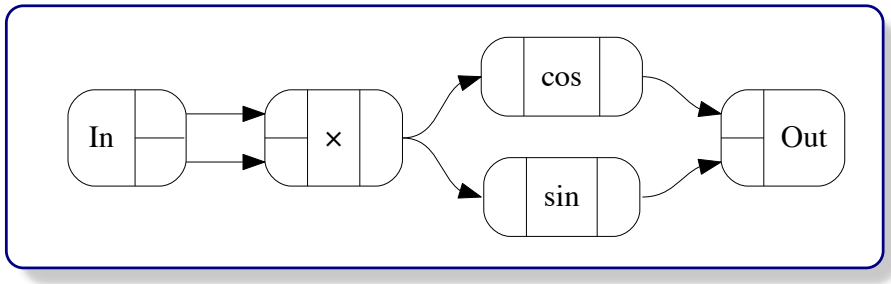


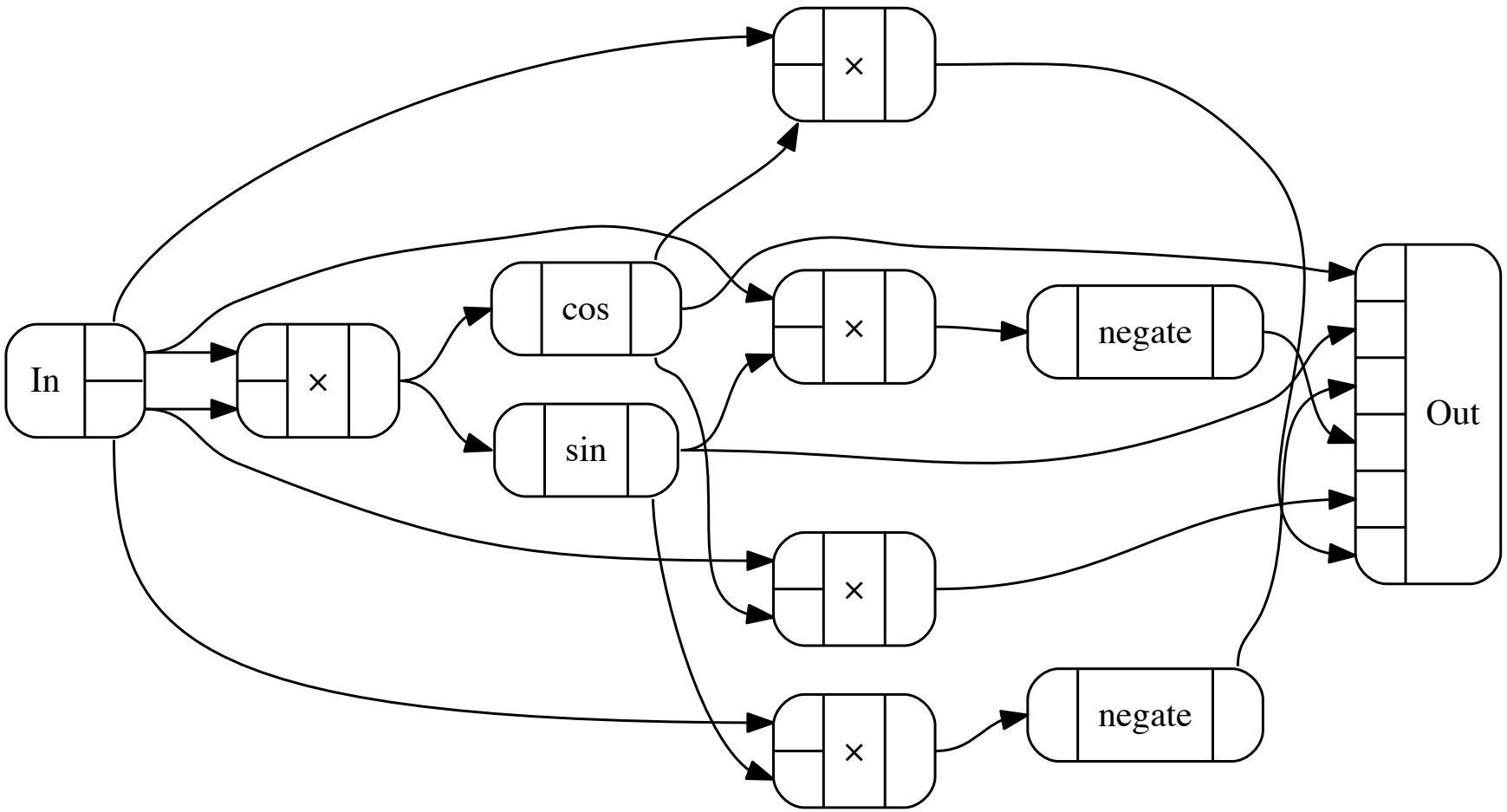
RAD example (dual function)



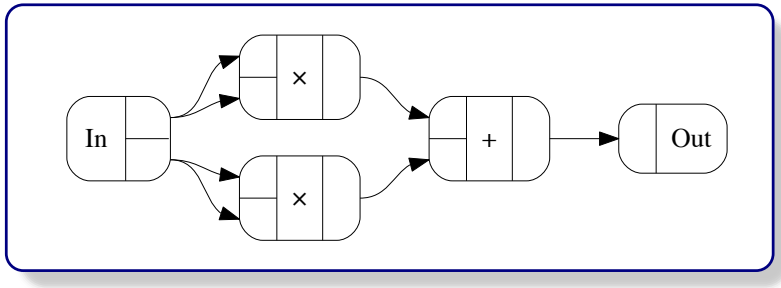


RAD example (matrix)

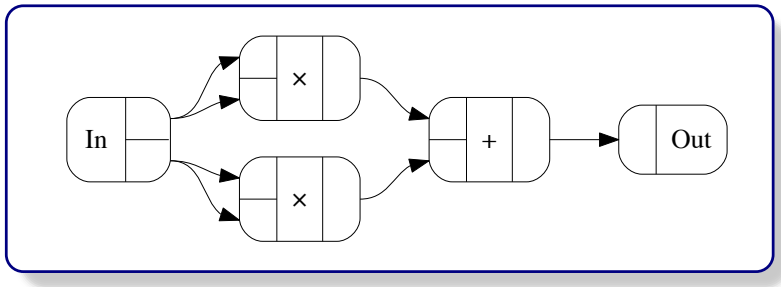


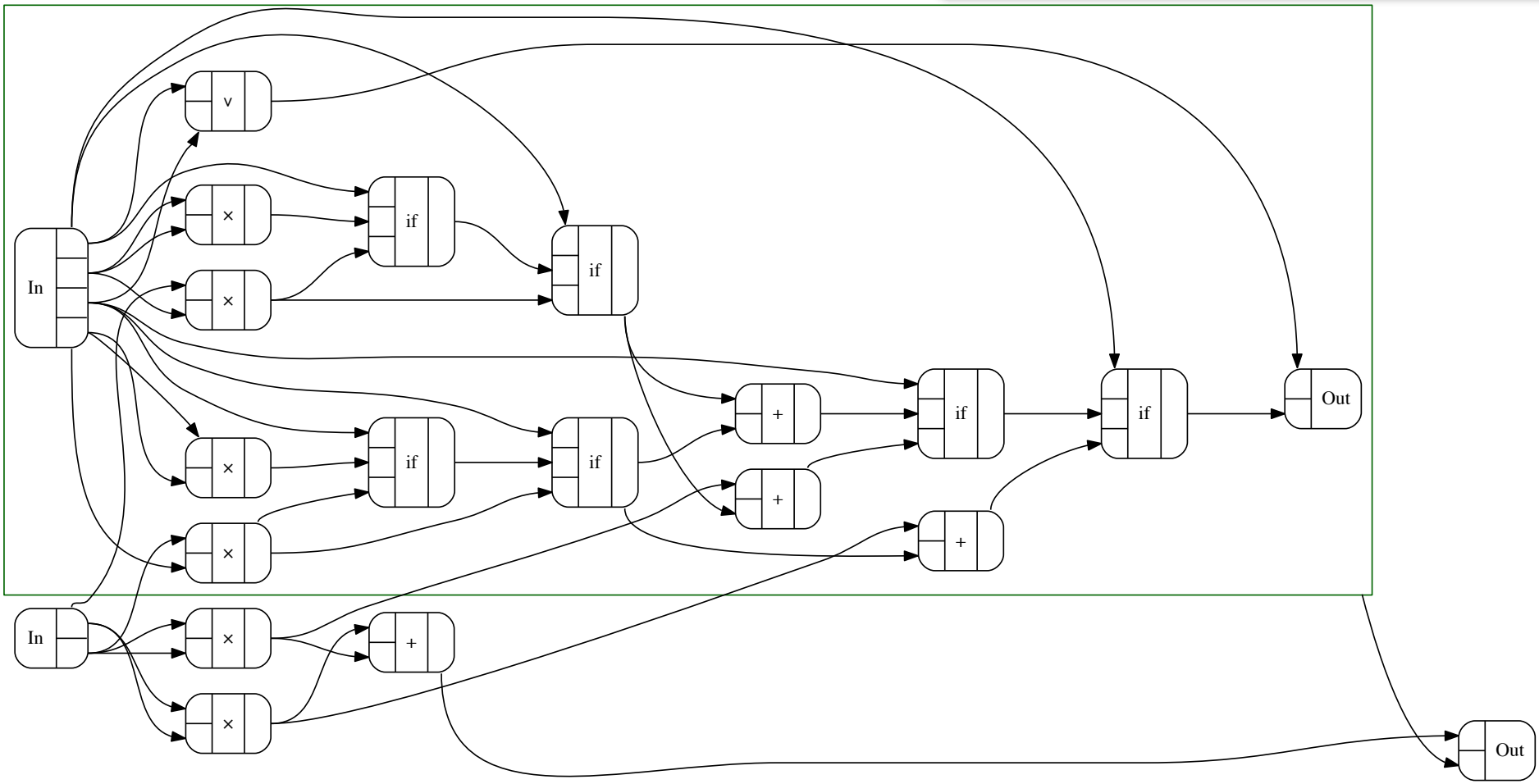


Incremental evaluation



Incremental evaluation





Conclusions

- Simple AD algorithm, specializing to forward, reverse, mixed.
- No graphs, tapes, tags, partial derivatives, or mutation.
- Parallel-friendly and low memory use.
- Calculated from simple, regular algebra problems.
- Generalizes to derivative categories other than linear maps.
- Differentiate regular Haskell code (via plugin).
- ICFP 2018 paper: pictures, proofs, incremental computation.

Reflections: recipe for success

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Key principles:

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- Focus on abstract notions, not specific representations.
- Calculate efficient implementation from simple specification.

Not previously applied to AD (afaik).

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Quandary: Most programming languages poor for function-like things.

Solution: *Compiling to categories*.

Symbolic vs automatic differentiation

Often described as opposing techniques:

- *Symbolic*:
  - Apply differentiation rules symbolically.
  - Can duplicate much work.
  - Needs algebraic manipulation.
- *Automatic*:
  - FAD: easy to implement but often inefficient.
  - RAD: efficient but tricky to implement.

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My view: *AD is SD done by a compiler.*

Compilers already work symbolically and preserve sharing.

Extra topics

- Examples: derivatives as functions
- Examples: reverse mode
- Incremental evaluation
- Compositional matrices