Circuit timing analysis, linear maps, and semantic morphisms

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Tabula

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Outline

Timing analysis

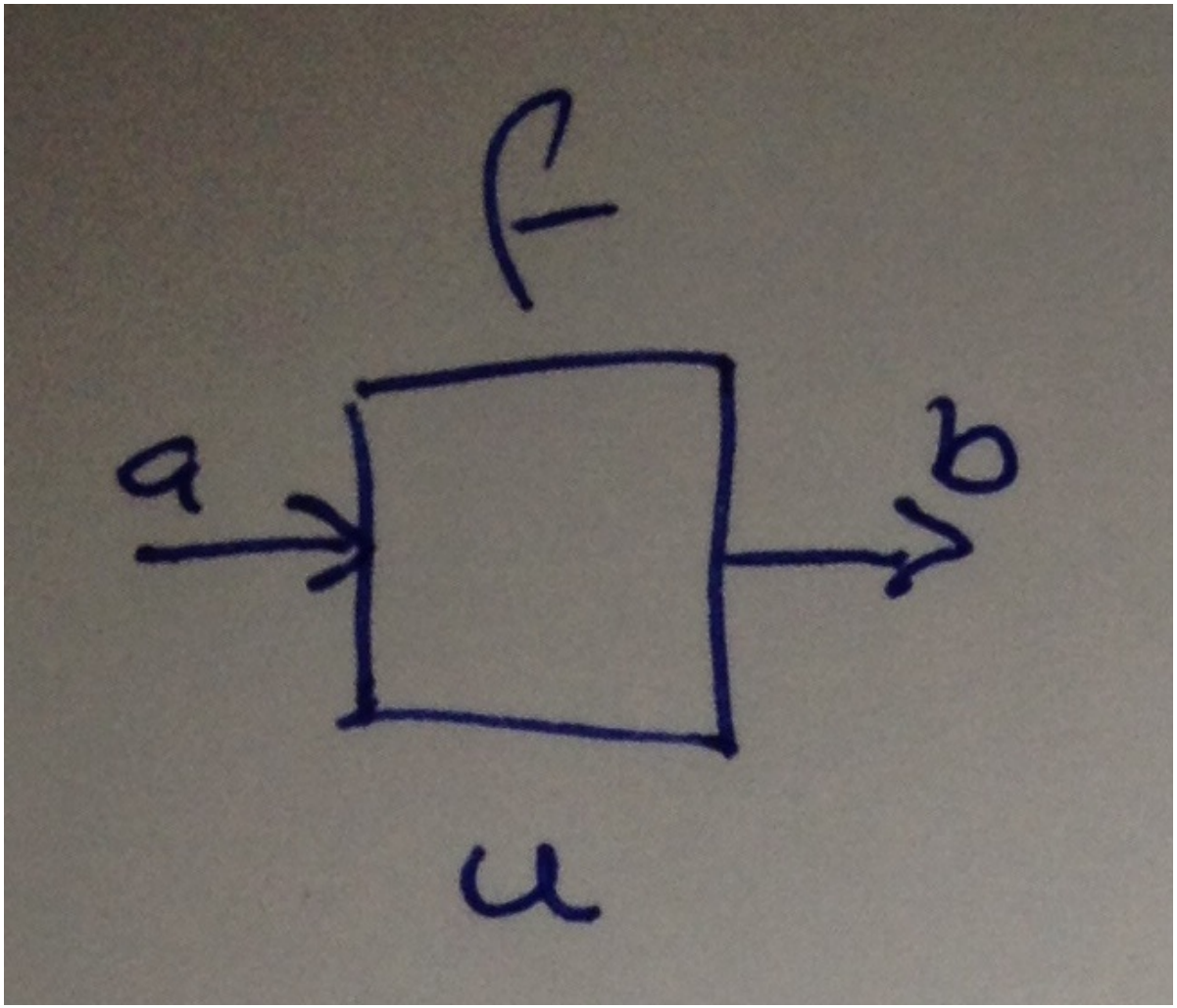
Linear transformations

Semantics and implementation

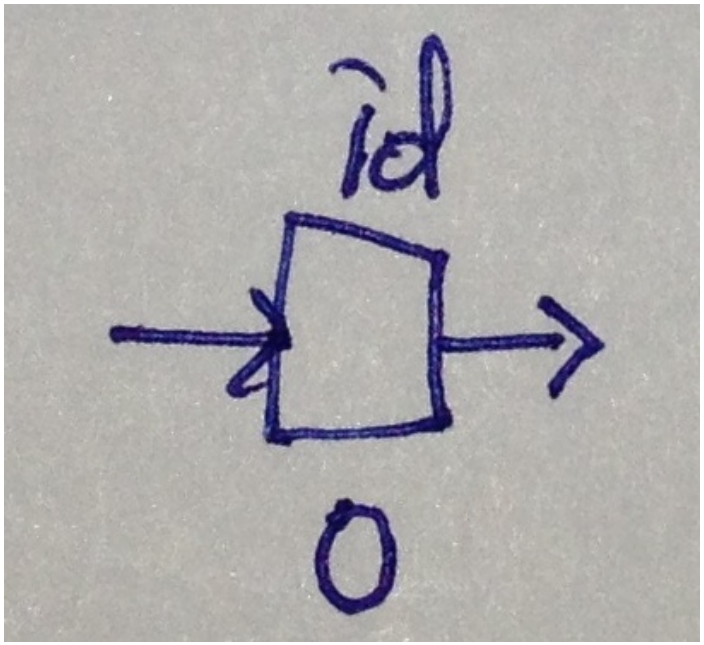
Timing analysis

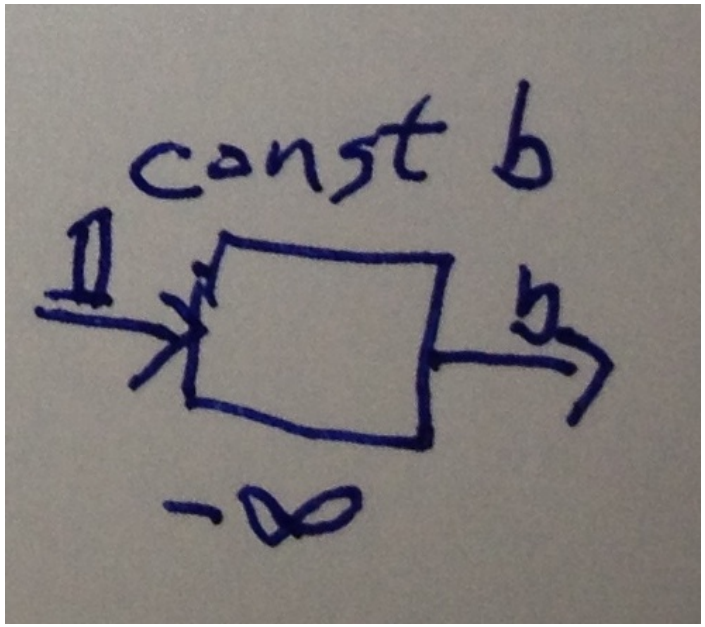
Simple timing analysis

Computation with a time delay:

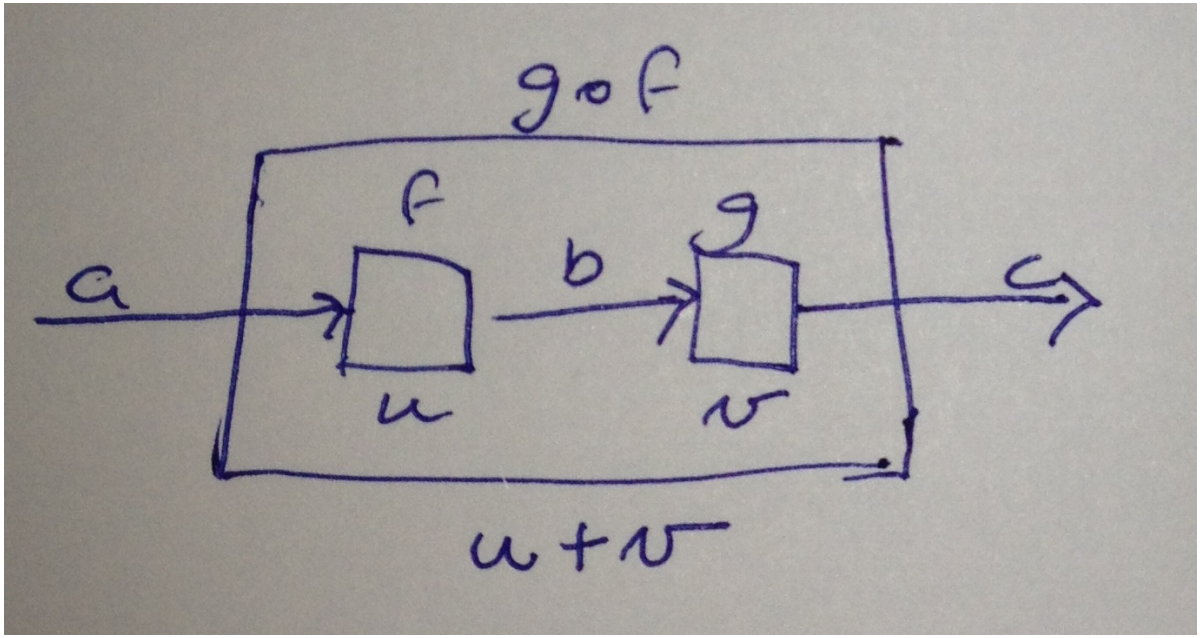


Trivial timings

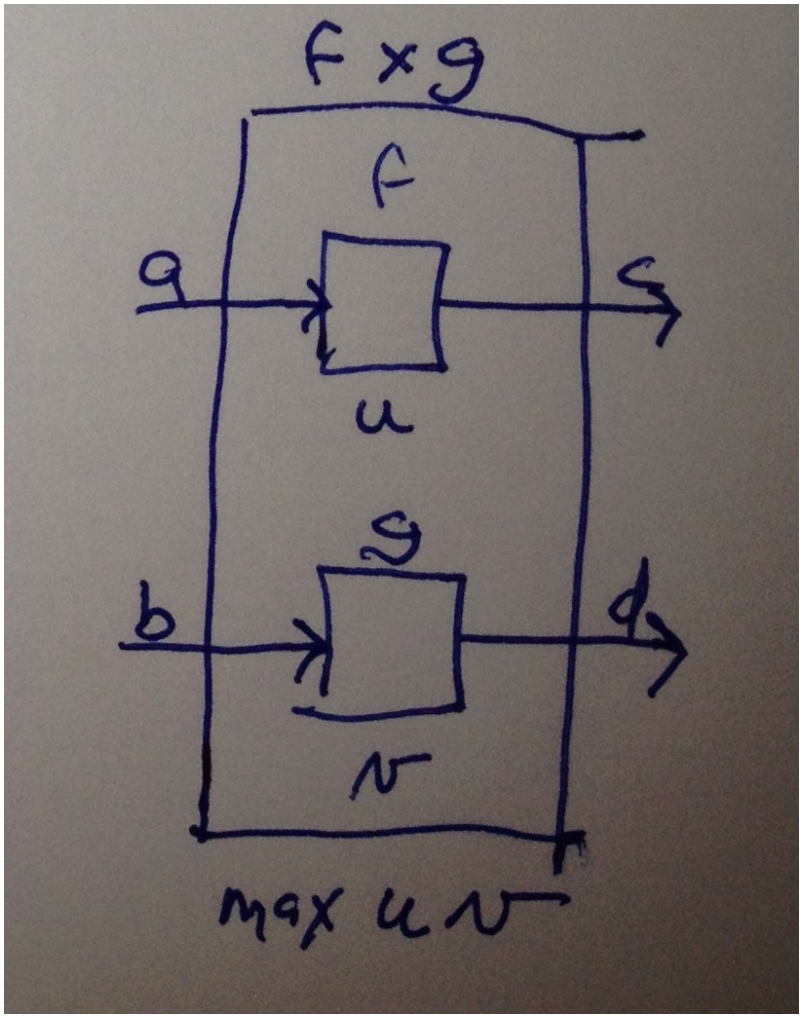




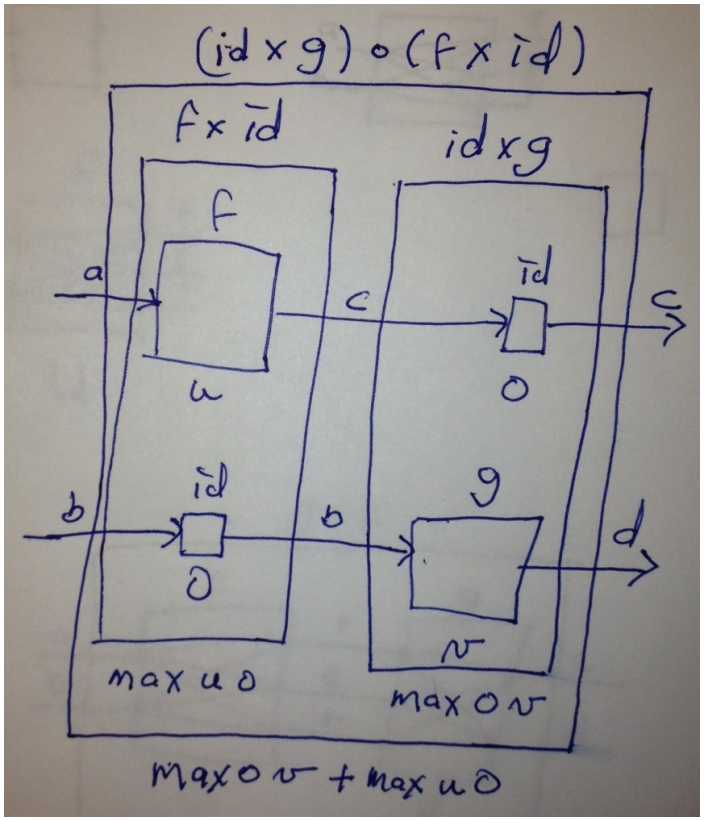
Sequential composition

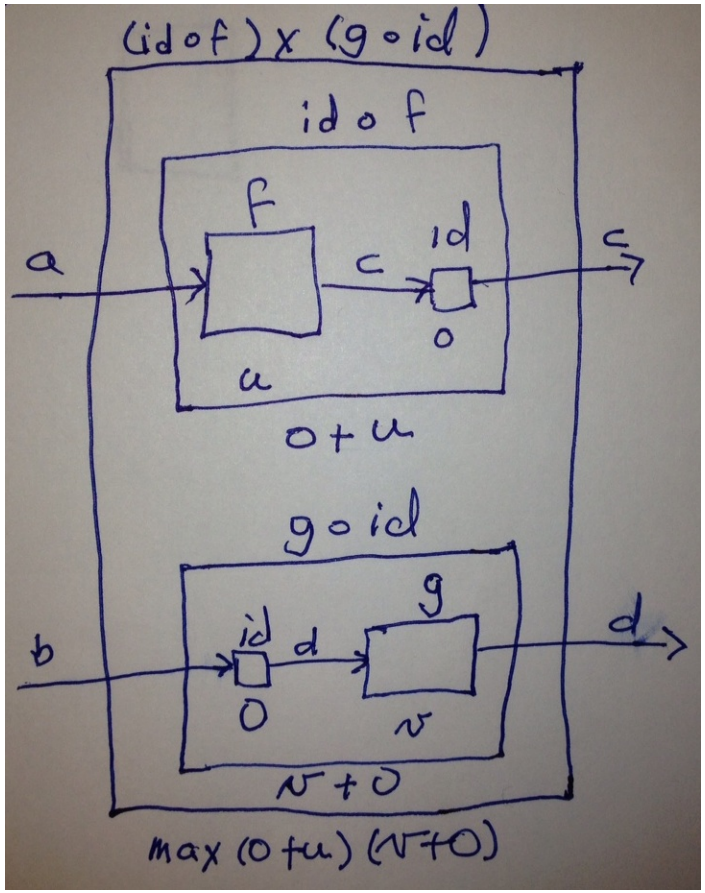


Parallel composition



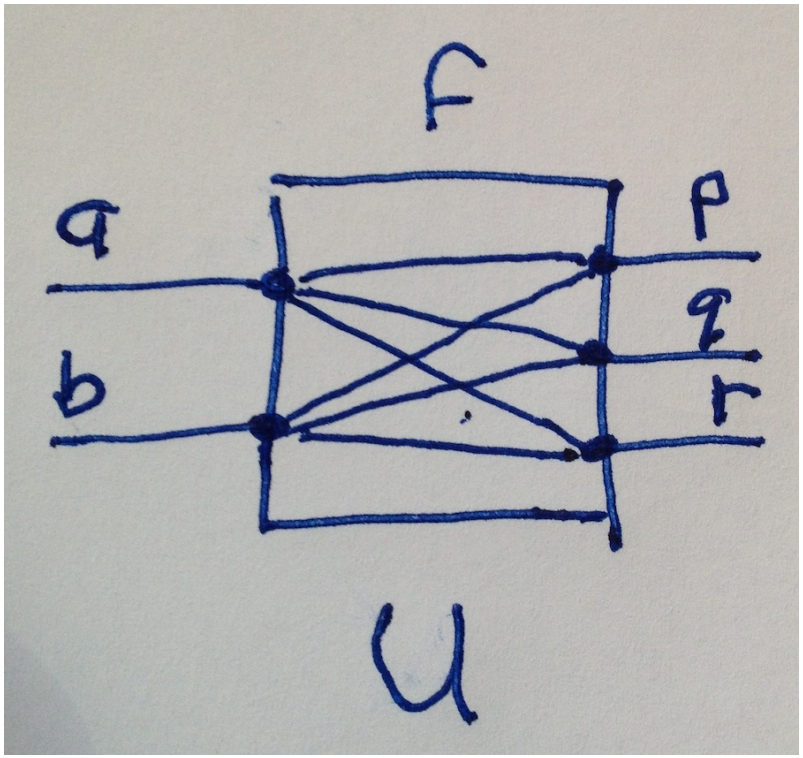
But ...





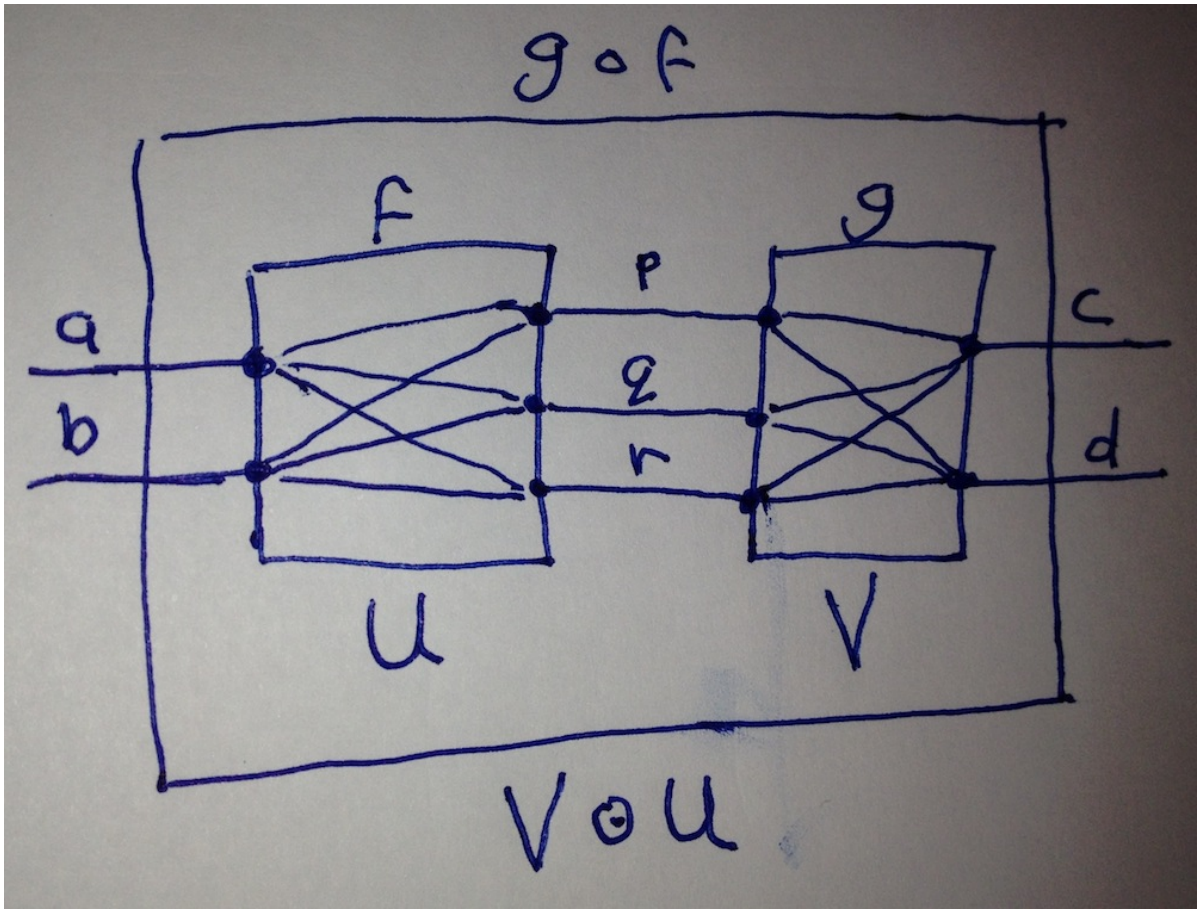
Oops: Same circuit ($f \times g$), different timings.

Multi-path analysis



- Max delay for *each* input/output pair
- How do delays *compose*?

How do delays *compose*?



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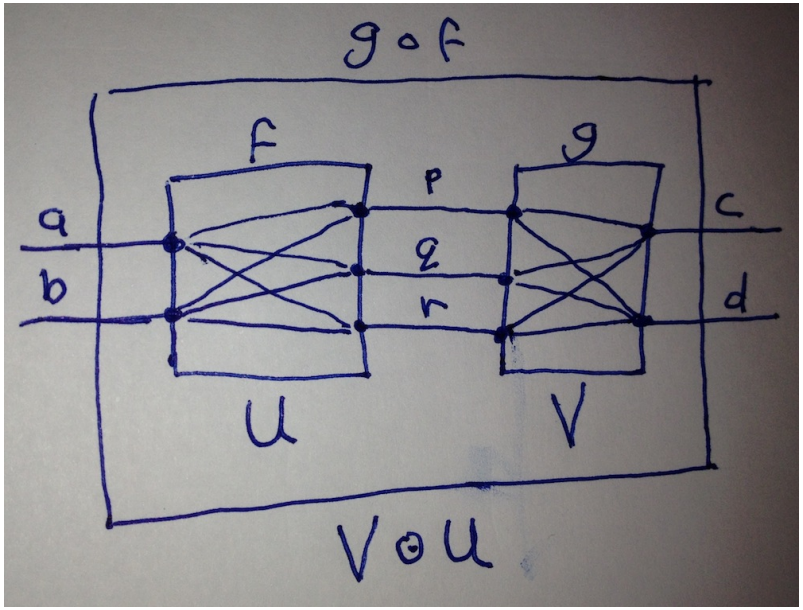

$V \odot U = W$, where

$$W_{i,k} = \operatorname*{Max}_j \left(U_{i,j} + V_{j,k} \right)$$

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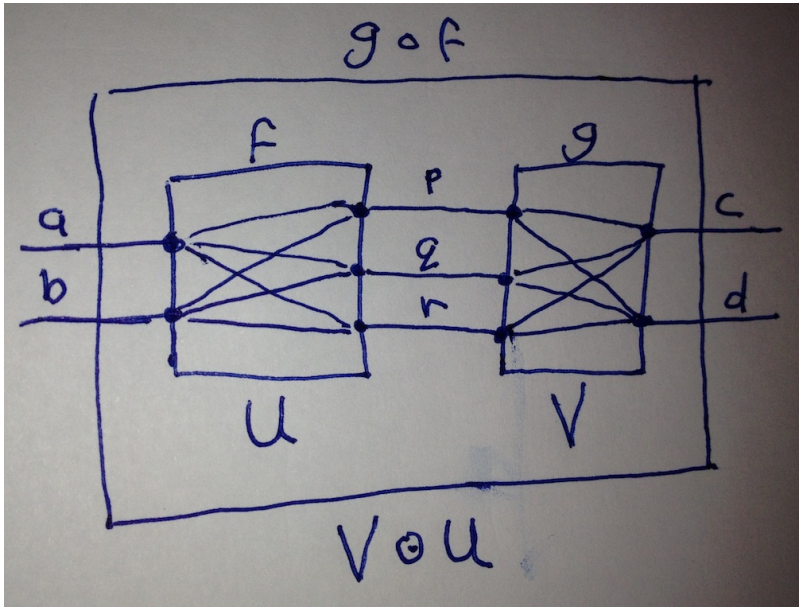

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Look familiar?

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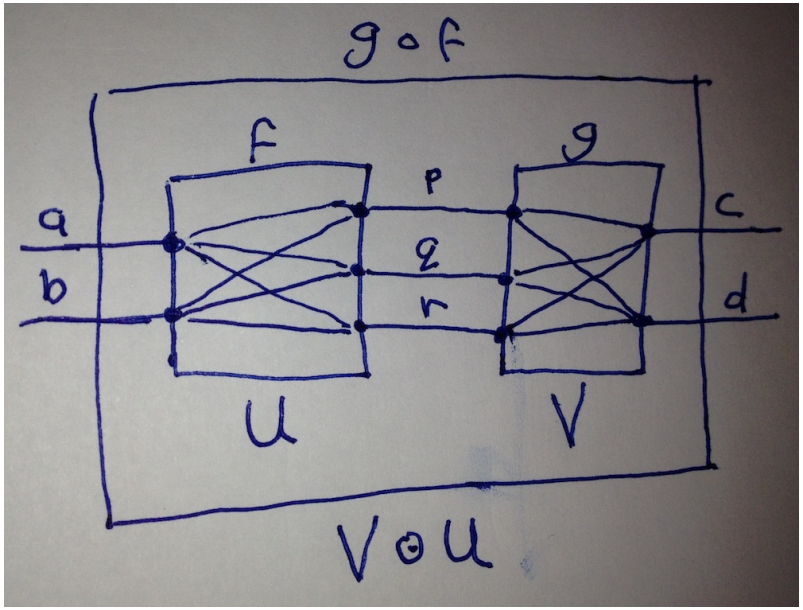

$V \odot U = W$, where

$$W_{i,k} = \operatorname*{Max}_j \left(U_{i,j} + V_{j,k}\right)$$

Look familiar? Matrix multiplication?

MaxPlus algebra

```
type Delay = MaxPlus Double
data MaxPlus a = MP a

instance Ord a => AdditiveGroup (MaxPlus a) where
  MP a ^+^ MP b = MP (a `max` b)
instance (Ord a, Num a) => VectorSpace (MaxPlus a) where
  type Scalar (MaxPlus a) = a
  a *^ MP b = MP (a + b)
```

Oops – We also need a zero.

VectorSpace is overkill. Module over a semi-ring suffices.

MaxPlus algebra

$$\textbf{type } \mathit{Delay} = \mathit{MaxPlus}\ \mathit{Double}$$

$$\textbf{data } \mathit{MaxPlus}\ a = -\infty \mid \mathit{Fi}\ a$$

$$\textbf{instance } \mathit{Ord}\ a \Rightarrow \mathit{AdditiveGroup}\ (\mathit{MaxPlus}\ a) \textbf{ where}$$

$$0 = -\infty$$

$$\mathit{MP}\ a \mathbin{\hat{+}} \mathit{MP}\ b = \mathit{MP}\ (a\ \text{`}\mathit{max}\text{`}\ b)$$

$$-\infty \mathbin{\hat{+}} \_ = -\infty$$

$$\_ \mathbin{\hat{+}} -\infty = -\infty$$

$$\textbf{instance } (\mathit{Ord}\ a, \mathit{Num}\ a) \Rightarrow \mathit{VectorSpace}\ (\mathit{MaxPlus}\ a) \textbf{ where}$$

$$\textbf{type } \mathit{Scalar}\ (\mathit{MaxPlus}\ a) = a$$

$$a \cdot \mathit{MP}\ b = \mathit{MP}\ (a + b)$$

$$\_ \cdot -\infty = -\infty$$

Linear transformations

Representation?

How might we represent linear maps/transformations $a \multimap b$?

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How might we represent linear maps/transformations $a \multimap b$?

- Matrices
- Functions
- What else?

Matrices

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$$

Static typing?

Statically sized matrices

$$\textbf{type}\ Mat\ m\ n\ a = Vec\ m\ (Vec\ n\ a)$$

$$\begin{aligned}&(\circ) :: (IsNat\ m, IsNat\ o) \Rightarrow \\ &\qquad Mat\ n\ o\ D \to Mat\ m\ n\ D \to Mat\ o\ m\ D \\ &no \circ mn = crossF\ dot\ (transpose\ no)\ mn\end{aligned}$$

$$\begin{aligned}&crossF :: (IsNat\ m, IsNat\ o) \Rightarrow \\ &\qquad (a \to b \to c) \to Vec\ o\ a \to Vec\ m\ b \to Mat\ o\ m\ c \\ &crossF\ f\ as\ bs = (\lambda a \to f\ a <\$> bs) <\$> as\end{aligned}$$

$$\begin{aligned}&dot :: (Ord\ a, Num\ a) \Rightarrow \\ &\qquad Vec\ n\ a \to Vec\ n\ a \to a \\ &u\ `dot`\ v = sum\ (zipWithV\ (*)\ u\ v)\end{aligned}$$

Generalizing

$$\textbf{type}\ \mathit{Mat}\ m\ n\ a = m\ (n\ a)$$

$$\begin{aligned}(\circ) :: (&\mathit{Functor}\ m, \mathit{Applicative}\ n, \mathit{Traversable}\ n, \mathit{Applicative}\ o) \Rightarrow \\ &\mathit{Mat}\ n\ o\ D \to \mathit{Mat}\ m\ n\ D \to \mathit{Mat}\ o\ m\ D \\ \mathit{no} \circ \mathit{mn} &= \mathit{crossF}\ \mathit{dot}\ (\mathit{sequenceA}\ \mathit{no})\ \mathit{mn}\end{aligned}$$

$$\begin{aligned}\mathit{crossF} :: (&\mathit{Functor}\ m, \mathit{Functor}\ o) \Rightarrow \\ &(a \to b \to c) \to o\ a \to m\ b \to \mathit{Mat}\ o\ m\ c \\ \mathit{crossF}\ f\ \mathit{as}\ \mathit{bs} &= (\lambda a \to f\ a <\$> \mathit{bs}) <\$> \mathit{as}\end{aligned}$$

$$\begin{aligned}\mathit{dot} :: (&\mathit{Foldable}\ n, \mathit{Applicative}\ n, \mathit{Ord}\ a, \mathit{Num}\ a) \Rightarrow \\ &n\ a \to n\ a \to a \\ u\ `\mathit{dot}`\ v &= \mathit{sum}\ (\mathit{liftA2}\ (*)\ u\ v)\end{aligned}$$

Represent via type family (old)

$$\textbf{class}\ \mathit{VectorSpace}\ v \Rightarrow \mathit{HasBasis}\ v\ \textbf{where}$$
$$\quad \textbf{type}\ \mathit{Basis}\ v :: *$$
$$\quad \mathit{coord} :: v \to (\mathit{Basis}\ v \to \mathit{Scalar}\ v)$$

Linear map as memoized function from basis:

$$\textbf{newtype}\ a \multimap b = L\ (\mathit{Basis}\ a \to^{M} b)$$

See *Beautiful differentiation* (ICFP 2009).

Represent as GADT

```
data a ⊸ b where
  Dot :: InnerSpace b ⇒
         b → (b ⊸ Scalar b)
  (:△) :: VS3 a c d ⇒    -- vector spaces with same scalar field
         (a ⊸ c) → (a ⊸ d) → (a ⊸ c × d)
```

Semantics and implementation

Semantics

$$[\![\cdot]\!] :: (a \multimap b) \to (a \to b)$$
$$[\![Dot\ b]\!] = dot\ b$$
$$[\![f :\vartriangle g]\!] = [\![f]\!] \vartriangle [\![g]\!]$$

where, on functions,

$$(f \vartriangle g)\ a = (f\ a, g\ a)$$

Recall:

data $a \multimap b$ **where**
$Dot :: InnerSpace\ b \Rightarrow b \to (b \multimap Scalar\ b)$
$(:\vartriangle) :: VS_3\ a\ c\ d \Rightarrow (a \multimap c) \to (a \multimap d) \to (a \multimap c \times d)$

Semantic type class morphisms

Category instance specification:

$$\begin{aligned} [\![id]\!] &\equiv id \\ [\![g \circ f]\!] &\equiv [\![g]\!] \circ [\![f]\!] \end{aligned}$$

Arrow instance specification:

$$\begin{aligned} [\![f \mathbin{\triangle} g]\!] &\equiv [\![f]\!] \mathbin{\triangle} [\![g]\!] \\ [\![f \times g]\!] &\equiv [\![f]\!] \times [\![g]\!] \end{aligned}$$

where

$$\begin{aligned} (\triangle) &:: Arrow\ (\rightsquigarrow) \Rightarrow (a \rightsquigarrow c) \to (a \rightsquigarrow d) \to (a \rightsquigarrow c \times d) \\ (\times) &:: Arrow\ (\rightsquigarrow) \Rightarrow (a \rightsquigarrow c) \to (b \rightsquigarrow d) \to (a \times b \rightsquigarrow c \times d) \end{aligned}$$

The *Category* and *Arrow* laws then follow.

Deriving a *Category* instance

One case:

$$
\begin{aligned}
& [\![(f :\vartriangle g) \circ h]\!] \\
\equiv\ & ([\![f]\!] \vartriangle [\![g]\!]) \circ [\![h]\!] \\
\equiv\ & [\![f]\!] \circ [\![h]\!] \vartriangle [\![g]\!] \circ [\![h]\!] \\
\equiv\ & [\![f \circ h \vartriangle g \circ h]\!]
\end{aligned}
$$

(where $f \circ h \vartriangle g \circ h \equiv (f \circ h) \vartriangle (g \circ h)$). Uses:

$$(f \vartriangle g) \circ h \equiv f \circ h \vartriangle g \circ h$$

Implementation:

$$(f :\vartriangle g) \circ h = f \circ h :\vartriangle g \circ h$$

Deriving a *Category* instance

$$
\begin{aligned}
& [\![Dot\ s \circ Dot\ b]\!] \\
\equiv\ & dot\ s \circ dot\ b \\
\equiv\ & dot\ (s \cdot b) \\
\equiv\ & [\![Dot\ (s \cdot b)]\!]
\end{aligned}
$$

$$
\begin{aligned}
& [\![Dot\ (a,b) \circ (f :\vartriangle g)]\!] \\
\equiv\ & dot\ (a,b) \circ ([\![f]\!] \vartriangle [\![g]\!]) \\
\equiv\ & add \circ (dot\ a \circ [\![f]\!] \vartriangle dot\ b \circ [\![g]\!]) \\
\equiv\ & dot\ a \circ [\![f]\!] \mathbin{\hat{+}} dot\ b \circ [\![g]\!] \\
\equiv\ & [\![Dot\ a \circ f \mathbin{\hat{+}} Dot\ b \circ g]\!]
\end{aligned}
$$

Uses:

$$
\begin{aligned}
dot\ (a,b) &\equiv add \circ (dot\ a \times dot\ b) \\
(k \times h) \circ (f \vartriangle g) &\equiv k \circ f \vartriangle h \circ g \\
[\![f \mathbin{\hat{+}} g]\!] &\equiv [\![f]\!] \mathbin{\hat{+}} [\![g]\!]
\end{aligned}
$$

Deriving an *Arrow* instance

$$
\begin{aligned}
& [\![f \vartriangle g]\!] \\
\equiv\ & [\![f]\!] \vartriangle [\![g]\!] \\
\equiv\ & [\![f :\vartriangle g]\!]
\end{aligned}
$$

$$
\begin{aligned}
& [\![f \times g]\!] \\
\equiv\ & [\![f]\!] \times [\![g]\!] \\
\equiv\ & [\![f]\!] \circ \mathit{fst} \vartriangle [\![g]\!] \circ \mathit{snd} \\
\equiv\ & [\![\mathit{compFst}\ f]\!] \vartriangle [\![\mathit{compSnd}\ g]\!] \\
\equiv\ & [\![\mathit{compFst}\ f :\vartriangle \mathit{compSnd}\ g]\!]
\end{aligned}
$$

assuming

$$
\begin{aligned}
[\![\mathit{compFst}\ \ f]\!] &\equiv [\![f]\!] \circ \mathit{fst} \\
[\![\mathit{compSnd}\ g]\!] &\equiv [\![g]\!] \circ \mathit{snd}
\end{aligned}
$$

Composing with *fst* and *snd*

$$compFst :: VS_3\ a\ b\ c \Rightarrow a \multimap c \to a \times b \multimap c$$
$$compSnd :: VS_3\ a\ b\ c \Rightarrow b \multimap c \to a \times b \multimap c$$

Derivation:

$$dot\ a \circ fst \equiv dot\ (a, 0)$$
$$(f \vartriangle g) \circ fst \equiv f \circ fst \vartriangle g \circ fst$$

Implementation:

$$compFst\ (Dot\ a) = Dot\ (a, 0)$$
$$compFst\ (f :\vartriangle g) = compFst\ f \vartriangle compFst\ g$$

$$compSnd\ (Dot\ b) = Dot\ (0, b)$$
$$compSnd\ (f :\vartriangle g) = compSnd\ f \vartriangle compSnd\ g$$

Adding linear maps

$$
\begin{aligned}
& [\![Dot\ b \mathbin{\hat{+}} Dot\ c]\!] \\
\equiv\ & dot\ b \mathbin{\hat{+}} dot\ c \\
\equiv\ & dot\ (b \mathbin{\hat{+}} c) \\
\equiv\ & [\![Dot\ (b \mathbin{\hat{+}} c)]\!]
\end{aligned}
$$

$$
\begin{aligned}
& [\![(f :\vartriangle g) \mathbin{\hat{+}} (h :\vartriangle k)]\!] \\
\equiv\ & ([\![f]\!] \vartriangle [\![g]\!]) \mathbin{\hat{+}} ([\![h]\!] \vartriangle [\![k]\!]) \\
\equiv\ & ([\![f]\!] \mathbin{\hat{+}} [\![h]\!]) \vartriangle ([\![g]\!] \mathbin{\hat{+}} [\![k]\!]) \\
\equiv\ & [\![(f \mathbin{\hat{+}} h) \vartriangle (g \mathbin{\hat{+}} k)]\!]
\end{aligned}
$$

Other cases don't type-check.

Uses (on functions):

$$(f \vartriangle g) \mathbin{\hat{+}} (h \vartriangle k) \equiv (f \mathbin{\hat{+}} h) \vartriangle (g \mathbin{\hat{+}} k)$$

What next?

- Fancier timing analysis
- What else is linear?
- More examples of semantic type class morphisms