Teaching new tricks to old programs

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Target Data Sciences

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Domain-specific embedded languages

New vocabularies, not new languages.

The Next 700 Programming Languages

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"... today ... 1,700 special programming languages used to 'communicate' in over 700 application areas."—*Computer Software* Issues, an American Mathematical Association Prospectus, July 1965.

Can we create fewer new vocabularies as well?

What does it mean?

$$x + 3 * y$$

It depends on x and y.

What does it mean?

$$\lambda x\ y \rightarrow x + 3 * y$$

It depends on $+$, $*$, and 3.

What does it mean?

$$\lambda x\ y \rightarrow x + 3 * y$$

It depends on $+$, $*$, and 3:

- *Int*, *Float*, *Double*
- $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$
- Vectors
- Polynomials
- Functions
- Regular expressions/languages
- Arbitrary rings, semirings, ...

Organizing interpretations

- Abstract algebra: interfaces and laws, e.g.,
  - Monoid, group, ring
  - Vector space
  - Functor, applicative, monad, foldable, traversable
  - Category, with products, with coproducts/sums
- Refactor and repurpose proofs and programs. (More with less.)

Example,

$$\mathit{fold} :: (\mathit{Foldable}\ f, \mathit{Monoid}\ m) \Rightarrow f\ m \rightarrow m$$

What does it mean?

$$\lambda x\ y \rightarrow x + 3 * y$$

- The most basic “operations”: λ, variables, and application.
- We can’t re-interpret/overload.
- What if there were a way?

Why overload lambda (etc)?

Same benefits as algebraic abstraction:

- Convenient notation.
- Generalized, principled interpretation.
- Modular programming and reasoning.

Why overload lambda?

- Convenient notation for functions.
- Alternative function implementations:
  - GPU code
  - Circuits
  - Javascript
- Enhanced functions:
  - Derivatives and integrals
  - Incremental evaluation
  - Interval analysis
  - Optimization
  - Root-finding
  - Constraint solving

How to overload lambda?

- Idea: eliminate it, and overload as usual.
- How?

Eliminating lambda

Introducing lambda

$$
\begin{aligned}
&\mathit{const} :: b \to (a \to b)\\
&\mathit{const}\ b = \lambda a \to b
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{id} :: a \to a\\
&\mathit{id} = \lambda a \to a
\end{aligned}
$$

$$
\begin{aligned}
&(\circ) :: (b \to c) \to (a \to b) \to (a \to c)\\
&g \circ f = \lambda a \to g\ (f\ a)
\end{aligned}
$$

$$
\begin{aligned}
&(\vartriangle) :: (a \to c) \to (a \to d) \to (a \to c \times d)\\
&f \vartriangle g = \lambda a \to (f\ a, g\ a)
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{curry} :: (a \times b \to c) \to (a \to b \to c)\\
&\mathit{curry}\ f = \lambda a \to \lambda b \to f\ (a, b)
\end{aligned}
$$

$$
\begin{aligned}
\mathit{apply} &:: (a \to b) \times a \to b\\
\mathit{apply} &= \lambda (f, a) \to f\ a\\
&= \mathit{uncurry}\ \mathit{id}
\end{aligned}
$$

Eliminating lambda

Systematically *un-inline:*

$$
\begin{array}{lll}
(\lambda p \rightarrow k) & \dashrightarrow & \mathit{const}\ k \\
(\lambda p \rightarrow p) & \dashrightarrow & \mathit{id} \\
(\lambda p \rightarrow u\ v) & \dashrightarrow & \mathit{apply} \circ ((\lambda p \rightarrow u) \vartriangle (\lambda p \rightarrow v)) \\
(\lambda p \rightarrow \lambda q \rightarrow u) & \dashrightarrow & \mathit{curry}\ (\lambda (p, q) \rightarrow u) \\
 & \dashrightarrow & \mathit{curry}\ (\lambda r \rightarrow u\ [p \coloneqq \mathit{fst}\ r, q \coloneqq \mathit{snd}\ r])
\end{array}
$$

Automate via a compiler plugin.

Examples

$$sqr :: Num\ a \Rightarrow a \rightarrow a$$
$$sqr\ a = a * a$$

$$magSqr :: Num\ a \Rightarrow a \times a \rightarrow a$$
$$magSqr\ (a, b) = sqr\ a + sqr\ b$$

$$cosSinProd :: Floating\ a \Rightarrow a \times a \rightarrow a \times a$$
$$cosSinProd\ (x, y) = (cos\ z, sin\ z)\ \textbf{where}\ z = x * y$$

After λ-elimination:

$$sqr = mulC \circ (id \vartriangle id)$$

$$magSqr = addC \circ (mulC \circ (exl \vartriangle exl) \vartriangle mulC \circ (exr \vartriangle exr))$$

$$cosSinProd = (cosC \vartriangle sinC) \circ mulC$$

Abstract algebra for functions

Interface:

$$\begin{array}{l}
\textbf{class}\ \mathit{Category}\ k\ \textbf{where} \\
\quad \mathit{id}\ \ :: a\ `k`\ a \\
\quad (\circ) :: (b\ `k`\ c) \to (a\ `k`\ b) \to (a\ `k`\ c) \\
\quad \textbf{infixr}\ 9\ \circ
\end{array}$$

Laws:

$$\begin{array}{ll}
\mathit{id} \circ f & \equiv f \\
g \circ \mathit{id} & \equiv g \\
(h \circ g) \circ f & \equiv h \circ (g \circ f)
\end{array}$$

Products

Interface:

$$
\begin{array}{l}
\textbf{class } \mathit{Category}\ k \Rightarrow \mathit{Cartesian}\ k\ \textbf{where} \\
\quad \textbf{type } a \times_k b \\
\quad \mathit{exl}\ :: (a \times_k b)\ \grave{}k\grave{}\ a \\
\quad \mathit{exr} :: (a \times_k b)\ \grave{}k\grave{}\ b \\
\quad (\vartriangle)\ :: (a\ \grave{}k\grave{}\ c) \rightarrow (a\ \grave{}k\grave{}\ d) \rightarrow (a\ \grave{}k\grave{}\ (c \times_k d)) \\
\quad \textbf{infixr } 3 \vartriangle
\end{array}
$$

Laws:

$$
\begin{array}{ll}
\mathit{exl} \circ (f \vartriangle g) & \equiv f \\
\mathit{exr} \circ (f \vartriangle g) & \equiv g \\
\mathit{exl} \circ h \vartriangle \mathit{exr} \circ h & \equiv h
\end{array}
$$

Coproducts

Dual to product.

$$
\begin{array}{l}
\textbf{class } \mathit{Category}\ k \Rightarrow \mathit{Cocartesian}\ k \textbf{ where} \\
\quad \textbf{type } a +_k b \\
\quad \mathit{inl}\ :: a \text{ `}k\text{` } (a +_k b) \\
\quad \mathit{inr} :: b \text{ `}k\text{` } (a +_k b) \\
\quad (\triangledown) :: (a \text{ `}k\text{` } c) \to (b \text{ `}k\text{` } c) \to ((a +_k b) \text{ `}k\text{` } c) \\
\quad \textbf{infixr } 2\ \triangledown
\end{array}
$$

Laws:

$$
\begin{array}{ll}
(f \triangledown g) \circ \mathit{inl} & \equiv f \\
(f \triangledown g) \circ \mathit{inr} & \equiv g \\
h \circ \mathit{inl} \triangledown h \circ \mathit{inr} & \equiv h
\end{array}
$$

Exponentials

First-class "functions" (morphisms):

$$
\begin{array}{l}
\textbf{class } \mathit{Cartesian}\ k \Rightarrow \mathit{Closed}\ k \textbf{ where} \\
\quad \textbf{type } a \Rightarrow_k b \\
\quad \mathit{apply} \quad :: ((a \Rightarrow_k b) \times_k a)\ \grave{}k\grave{}\ b \\
\quad \mathit{curry} \quad :: ((a \times_k b)\ \grave{}k\grave{}\ c) \rightarrow (a\ \grave{}k\grave{}\ (b \Rightarrow_k c)) \\
\quad \mathit{uncurry} :: (a\ \grave{}k\grave{}\ (b \Rightarrow_k c)) \rightarrow ((a \times_k b)\ \grave{}k\grave{}\ c)
\end{array}
$$

Laws:

$$
\begin{array}{ll}
\mathit{uncurry}\ (\mathit{curry}\ f) & \equiv f \\
\mathit{curry}\ (\mathit{uncurry}\ g) & \equiv g \\
\mathit{apply} \circ (\mathit{curry}\ f \circ \mathit{exl} \vartriangle \mathit{exr}) & \equiv f
\end{array}
$$

Misc operations

$$
\begin{array}{l}
\textbf{class}\ \mathit{NumCat}\ k\ a\ \textbf{where} \\
\quad \begin{array}{ll}
\mathit{negateC} & :: a\ \grave{}k\grave{}\ a \\
\mathit{addC}, \mathit{sub}, \mathit{mulC} & :: (a \times_k a)\ \grave{}k\grave{}\ a \\
\ldots &
\end{array}
\end{array}
$$

...

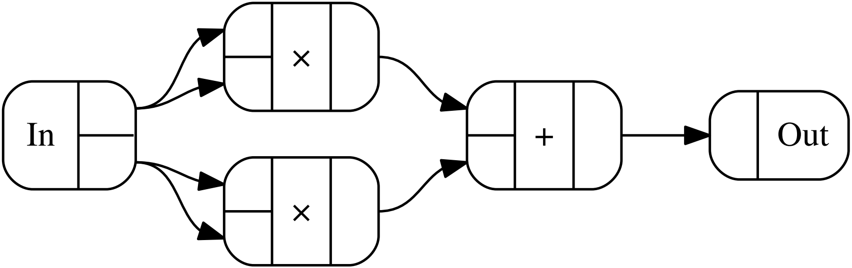
Changing interpretations

- We've eliminated lambdas and variables
- and replaced them with an algebraic vocabulary.
- What happens if we *replace* $(\rightarrow)$ *with other instances?*
  (Via compiler plugin.)

Computation graphs — example

$$magSqr\ (a, b) = sqr\ a + sqr\ b$$

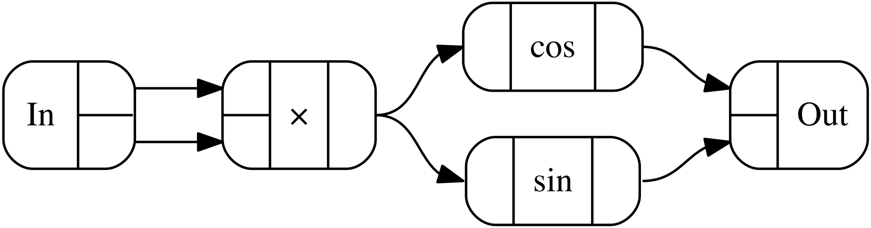
$$magSqr = addC \circ (mulC \circ (exl \vartriangle exl) \vartriangle mulC \circ (exr \vartriangle exr))$$



Computation graphs — example

$$cosSinProd\ (x, y) = (cos\ z, sin\ z)\ \textbf{where}\ z = x * y$$

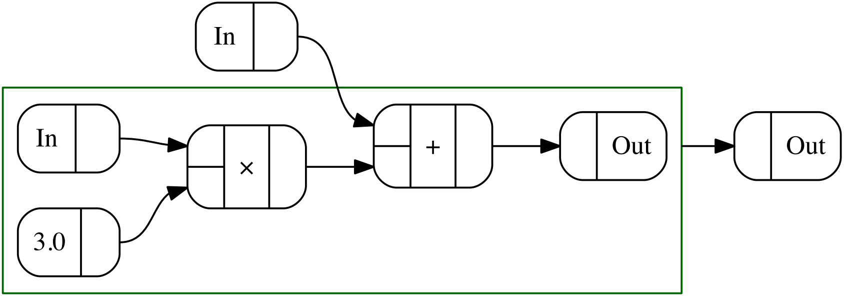
$$cosSinProd = (cosC \vartriangle sinC) \circ mulC$$



Computation graphs — example

$\lambda x\ y \rightarrow x + 3 * y$

$curry\ (addC \circ (exl \vartriangle mulC \circ (const\ 3.0 \vartriangle exr)))$



Computation graphs — implementation sketch

$$\textbf{newtype}\ \mathit{Graph}\ a\ b = \mathit{Graph}\ (\mathit{Ports}\ a \rightarrow \mathit{GraphM}\ (\mathit{Ports}\ b))$$

$$\textbf{type}\ \mathit{GraphM} = \mathit{State}\ (\mathit{PortNum}, [\mathit{Comp}])$$

$$\textbf{data}\ \mathit{Comp} = \forall a\ b . \mathit{Comp}\ (\mathit{Template}\ a\ b)\ (\mathit{Ports}\ a)\ (\mathit{Ports}\ b)$$

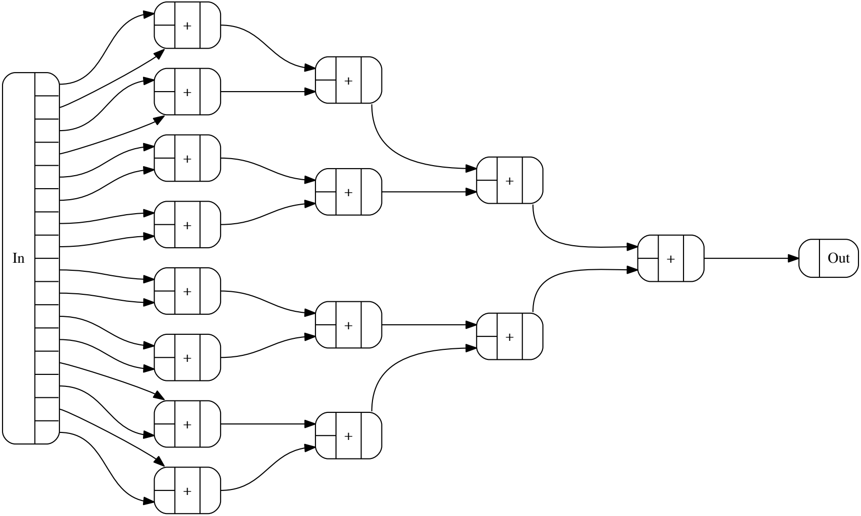
$$\begin{array}{l}
\textbf{data}\ \mathit{Template} :: * \rightarrow * \rightarrow *\ \textbf{where} \\
\quad \mathit{Prim} \quad\quad :: \mathit{String} \rightarrow \mathit{Template}\ a\ b \\
\quad \mathit{Subgraph} :: \mathit{Graph}\ a\ b \rightarrow \mathit{Template}\ ()\ (a \rightarrow b)
\end{array}$$

$$\begin{array}{l}
\textbf{instance}\ \mathit{Category}\ \mathit{Graph}\ \textbf{where} \\
\quad \mathit{id} = \mathit{Graph}\ \mathit{return} \\
\quad \mathit{Graph}\ g \circ \mathit{Graph}\ f = \mathit{Graph}\ (g \mathrel{<\!\!=\!\!<} f)
\end{array}$$

$$\begin{array}{l}
\textbf{instance}\ \mathit{BoolCat}\ \mathit{Graph}\ \textbf{where} \\
\quad \mathit{notC} \ = \mathit{genComp}\ \texttt{"}\neg\texttt{"} \\
\quad \mathit{andC} = \mathit{genComp}\ \texttt{"}\wedge\texttt{"} \\
\quad \mathit{orC} \quad = \mathit{genComp}\ \texttt{"}\vee\texttt{"}
\end{array}$$

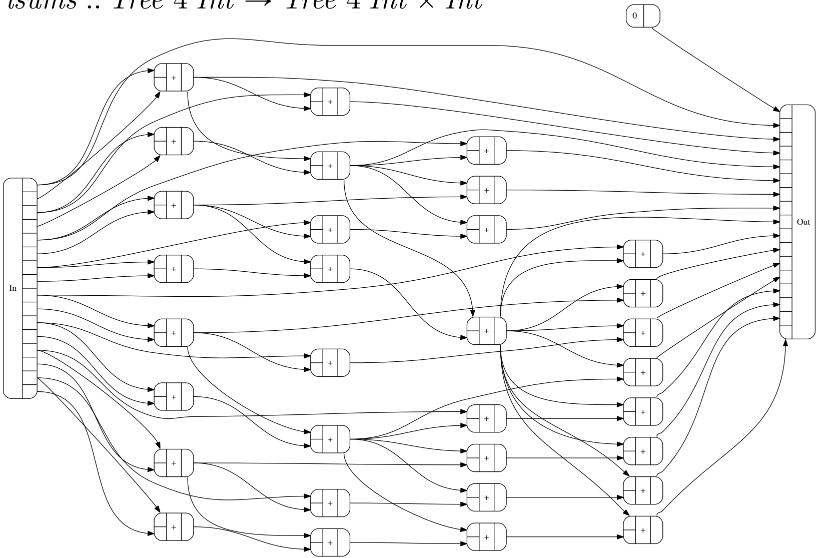
Computation graphs — fold

$sum :: Tree\ 4\ Int \rightarrow Int$



Computation graphs — scan

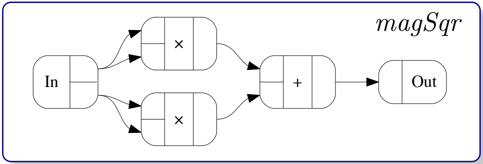
$lsums :: Tree\ 4\ Int \rightarrow Tree\ 4\ Int \times Int$



Haskell to hardware

Convert graphs to Verilog:

```
module magSqr (In_0, In_1, Out);
  input [31:0] In_0;
  input [31:0] In_1;
  output [31:0] Out;
  wire [31:0] Plus_I0;
  wire [31:0] Times_I3;
  wire [31:0] Times_I4;
  assign Plus_I0 = Times_I3 + Times_I4;
  assign Out = Plus_I0;
  assign Times_I3 = In_0 * In_0;
  assign Times_I4 = In_1 * In_1;
endmodule
```

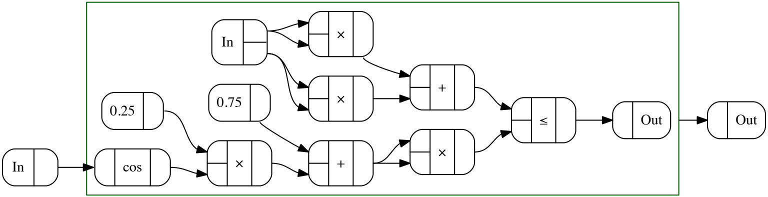


Example — graphics

$disk :: \mathbb{R} \to Region$
$disk\ r\ p = magSqr\ p \leqslant sqr\ r$

$woob\ t = disk\ (0.75 + 0.25 * cos\ t)$

type $Region = \mathbb{R} \times \mathbb{R} \to Bool$




```
bool uwoob (float in0, float in1, float in2)  // Generated GLSL
{ float v17 = 1.0;
  float v23 = v17 / (0.75 + 0.25 * cos (in0));
  float v24 = in1 * v23;
  float v26 = in2 * v23;
  return v24 * v24 + v26 * v26 <= v17;
}
vec4 effect (vec2 p) { return bw(uwoob(time,p.x,p.y)); }
```

Automatic differentiation

$\textbf{data}\ D\ a\ b = D\ (a \to b \times (a \multimap b))$ -- Derivatives are linear maps.

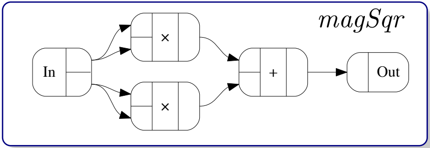
$$linearD\ f = D\ (\lambda a \to (f\ a, linear\ f))$$

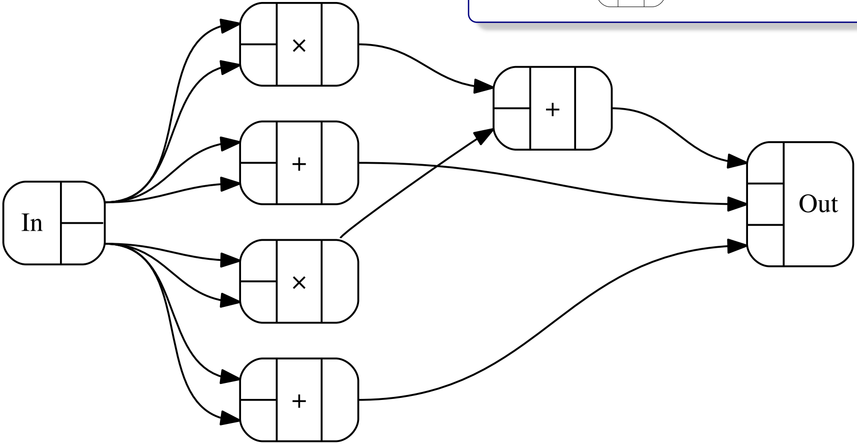
$\textbf{instance}\ Category\ D\ \textbf{where}$
$\quad id = linearD\ id$
$\quad D\ g \circ D\ f = D\ (\lambda a \to \textbf{let}\ \{(b, f') = f\ a; (c, g') = g\ b\}\ \textbf{in}\ (c, g' \circ f'))$

$\textbf{instance}\ Cartesian\ D\ \textbf{where}$
$\quad exl = linearD\ exl$
$\quad exr = linearD\ exr$
$\quad D\ f \vartriangle D\ g = D\ (\lambda a \to \textbf{let}\ \{(b, f') = f\ a; (c, g') = g\ a\}\ \textbf{in}\ ((b, c), f' \vartriangle g'))$

$\textbf{instance}\ NumCat\ D\ \textbf{where}$
$\quad negateC = linearD\ negateC$
$\quad addC = linearD\ addC$
$\quad mulC = D\ (mulC \vartriangle \lambda(a, b) \to linear\ (\lambda(da, db) \to da * b + db * a))$

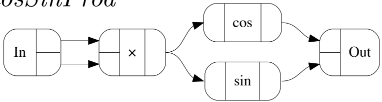
Composing interpretations (*Graph* and *D*)

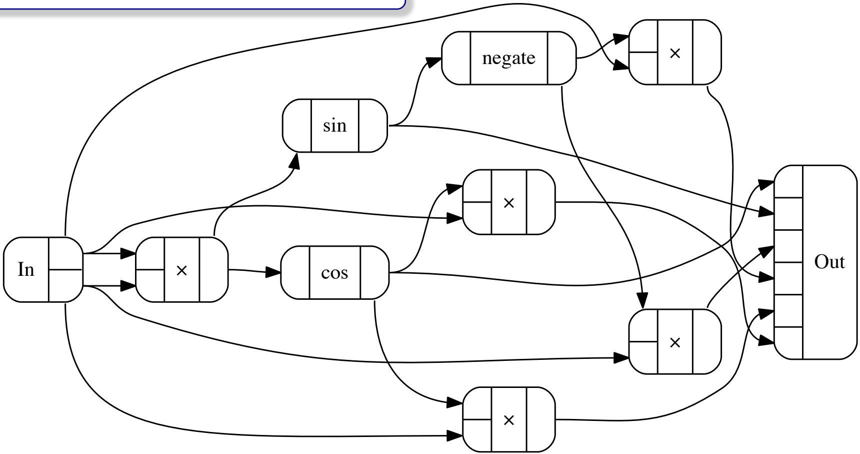




Composing interpretations (*Graph* and *D*)

cosSinProd





Interval analysis

$$\mathbf{data}\ \mathit{IFun}\ a\ b = \mathit{IFun}\ (\mathit{Interval}\ a \rightarrow \mathit{Interval}\ b)$$

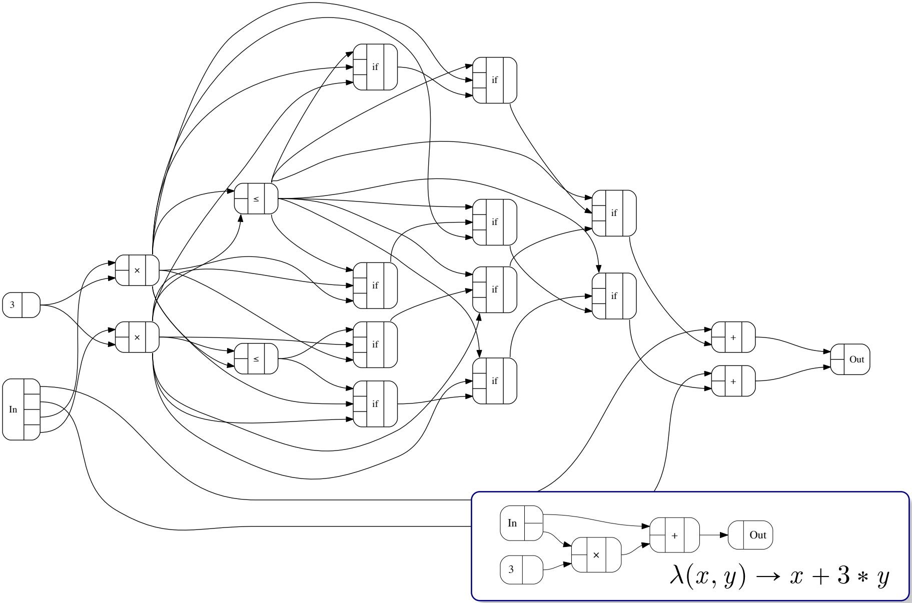
$$
\begin{array}{l}
\mathbf{type}\ \mathit{family}\ \mathit{Interval}\ a \\
\mathbf{type\ instance}\ \mathit{Interval}\ \mathit{Double} \ \ = \mathit{Double} \times \mathit{Double} \\
\mathbf{type\ instance}\ \mathit{Interval}\ (a \times b) = \mathit{Interval}\ a \times \mathit{Interval}\ b \\
\mathbf{type\ instance}\ \mathit{Interval}\ (a \rightarrow b) = \mathit{Interval}\ a \rightarrow \mathit{Interval}\ b
\end{array}
$$

$$
\begin{array}{l}
\mathbf{instance}\ \mathit{Category}\ \mathit{IFun}\ \mathbf{where} \\
\quad \mathit{id} = \mathit{IFun}\ \mathit{id} \\
\quad \mathit{IFun}\ g \circ \mathit{IFun}\ f = \mathit{IFun}\ (g \circ f) \\
\ldots
\end{array}
$$

$$
\begin{array}{l}
\mathbf{instance}\ \mathit{Cartesian}\ \mathit{IFun}\ \mathbf{where} \\
\quad \mathit{exl} = \mathit{IFun}\ \mathit{exl} \\
\quad \mathit{exr} = \mathit{IFun}\ \mathit{exr} \\
\quad \mathit{IFun}\ f \vartriangle \mathit{IFun}\ g = \mathit{IFun}\ (f \vartriangle g)
\end{array}
$$

$$
\begin{array}{l}
\mathbf{instance}\ (\mathit{Interval}\ a \sim (a \times a), \mathit{Num}\ a, \mathit{Ord}\ a) \Rightarrow \mathit{NumCat}\ \mathit{IFun}\ a\ \mathbf{where} \\
\quad \mathit{addC} = \mathit{IFun}\ (\lambda((a_{lo}, a_{hi}), (b_{lo}, b_{hi})) \rightarrow (a_{lo} + b_{lo}, a_{hi} + b_{hi})) \\
\quad \mathit{mulC} = \mathit{IFun}\ (\lambda((a_{lo}, a_{hi}), (b_{lo}, b_{hi})) \rightarrow \\
\qquad \mathit{minmax}\ [a_{lo} * b_{lo}, a_{lo} * b_{hi}, a_{hi} * b_{lo}, a_{hi} * b_{hi}] \\
\quad \ldots
\end{array}
$$

Interval analysis — example



Constraint solving (with John Wiegley)

$$
\begin{aligned}
&\textbf{newtype}\ \mathit{SMT}\ a\ b = \mathit{SMT}\ (\mathit{Kleisli}\ \mathit{Z3}\ (E\ a)\ (E\ b))\\[1ex]
&\textbf{data}\ E :: * \to * \ \textbf{where}\\
&\quad \mathit{PrimE} :: \mathit{AST} \to E\ a\\
&\quad \mathit{PairE}\ \ :: E\ a \to E\ b \to E\ (a \times b)\\[1ex]
&\textbf{instance}\ \mathit{Category}\ \mathit{SMT}\ \textbf{where}\\
&\quad \mathit{id} = \mathit{SMT}\ \mathit{id}\\
&\quad \mathit{SMT}\ g \circ \mathit{SMT}\ f = \mathit{SMT}\ (g \circ f)\\[1ex]
&\textbf{instance}\ \mathit{Cartesian}\ \mathit{SMT}\ \textbf{where}\\
&\quad \mathit{exl}\ \ = \mathit{SMT}\ (\mathit{arr}\ (\mathit{exl} \circ \mathit{unpairE}))\\
&\quad \mathit{exr} = \mathit{SMT}\ (\mathit{arr}\ (\mathit{exr} \circ \mathit{unpairE}))\\
&\quad \mathit{SMT}\ f \vartriangle \mathit{SMT}\ g = \mathit{SMT}\ (\mathit{arr}\ \mathit{PairE} \circ (f \vartriangle g))\\[1ex]
&\textbf{instance}\ \mathit{Num}\ a \Rightarrow \mathit{NumCat}\ \mathit{SMT}\ a\ \textbf{where}\\
&\quad \mathit{negateC} = \mathit{liftE}_1\ \mathit{mkUnaryMinus}\\
&\quad \mathit{addC}\ \ \ \ \ = \mathit{liftE}_2\ \mathit{mkAdd}\\
&\quad \mathit{subC}\ \ \ \ \ \ = \mathit{liftE}_2\ \mathit{mkSub}\\
&\quad \mathit{mulC}\ \ \ \ \ = \mathit{liftE}_2\ \mathit{mkMul}
\end{aligned}
$$

Constraint solving (with John Wiegley)

$$
\begin{array}{l}
\mathit{pred} :: (\mathit{Num}\ a, \mathit{Ord}\ a) \Rightarrow a \times a \rightarrow \mathit{Bool} \\
\mathit{pred}\ (x, y) = \\
\quad x < y\ \wedge \\
\quad y < 100\ \wedge \\
\quad 0 \leqslant x - 3 + 7 * y\ \wedge \\
\quad (x \equiv y\ \vee\ y + 20 \equiv x + 30)
\end{array}
$$

Solution: $(-8, 2)$.

Other examples

- Linear maps
- Incremental evaluation
- Polynomials
- Nondeterministic and probabilistic programming

Domain-specific embedded languages (DSELs)

- *Shallow* (just a library):
  - Great fit with host language.
  - Easy to implement and use.
  - Hard to optimize.
  - Good choice for *expressing ideas*.
- *Deep* (syntactic representation):
  - More room for analysis and optimization.
  - Harder to implement; redundant with host compiler.
  - Less semantic guidance.
  - Syntactically awkward in places.
  - Good choice for *efficient implementation*.
- *Compiling to categories*:
  - Great fit with host language.
  - Semantic guidance.
  - Easy to implement.
  - Analysis, optimization, non-standard target architectures.

For more details

- The paper *Compiling to categories* (February 2017)
- GitHub project page